1 Trigonometric Identities you must remember

The “big three” trigonometric identities are

\[ \sin^2 t + \cos^2 t = 1 \]  \hspace{1cm} (1)
\[ \sin(A + B) = \sin A \cos B + \cos A \sin B \]  \hspace{1cm} (2)
\[ \cos(A + B) = \cos A \cos B - \sin A \sin B \]  \hspace{1cm} (3)

Using these we can derive many other identities. Even if we commit the other useful identities to memory, these three will help be sure that our signs are correct, etc.

2 Two more easy identities

From equation (1) we can generate two more identities. First, divide each term in (1) by \( \cos^2 t \) (assuming it is not zero) to obtain

\[ \tan^2 t + 1 = \sec^2 t. \]  \hspace{1cm} (4)

When we divide by \( \sin^2 t \) (again assuming it is not zero) we get

\[ 1 + \cot^2 t = \csc^2 t. \]  \hspace{1cm} (5)

3 Identities involving the difference of two angles

From equations (2) and (3) we can get several useful identities. First, recall that

\[ \cos(-t) = \cos t, \quad \sin(-t) = -\sin t. \]  \hspace{1cm} (6)

From (2) we see that

\[ \sin(A - B) = \sin(A + (-B)) = \sin A \cos(-B) + \cos A \sin(-B) \]

which, using the relationships in (6), reduces to

\[ \sin(A - B) = \sin A \cos B - \cos A \sin B. \]  \hspace{1cm} (7)

In a similar way, we can use equation (3) to find

\[ \cos(A - B) = \cos(A + (-B)) = \cos A \cos(-B) - \sin A \sin(-B) \]

which simplifies to

\[ \cos(A - B) = \cos A \cos B + \sin A \sin B. \]  \hspace{1cm} (8)

Notice that by remembering the identities (2) and (3) you can easily work out the signs in these last two identities.
4 Identities involving products of sines and cosines

If we now add equation (2) to equation (7)

\[ \sin(A-B) = \sin A \cos B - \cos A \sin B \]
\[ + (\sin(A+B) = \sin A \cos B + \cos A \sin B) \]

we find

\[ \sin(A-B) + \sin(A+B) = 2 \sin A \cos B \]

and dividing both sides by 2 we obtain the identity

\[ \sin A \cos B = \frac{1}{2} \sin(A-B) + \frac{1}{2} \sin(A+B). \] (9)

In the same way we can add equations (3) and (8)

\[ \cos(A-B) = \cos A \cos B + \sin A \sin B \]
\[ + (\cos(A+B) = \cos A \cos B - \sin A \sin B) \]

to get

\[ \cos(A-B) + \cos(A+B) = 2 \cos A \cos B \]

which can be rearranged to yield the identity

\[ \cos A \cos B = \frac{1}{2} \cos(A-B) + \frac{1}{2} \cos(A+B). \] (10)

Suppose we wanted an identity involving \( \sin A \sin B \). We can find one by slightly modifying the last thing we did. Rather than adding equations (3) and (8), all we need to do is subtract equation (3) from equation (8):

\[ \cos(A-B) = \cos A \cos B + \sin A \sin B \]
\[ - (\cos(A+B) = \cos A \cos B - \sin A \sin B) \]

This gives

\[ \cos(A-B) - \cos(A+B) = 2 \sin A \sin B \]

or, in the form we prefer,

\[ \sin A \sin B = \frac{1}{2} \cos(A-B) - \frac{1}{2} \cos(A+B). \] (11)

5 Double angle identities

Now a couple of easy ones. If we let \( A = B \) in equations (2) and (3) we get the two identities

\[ \sin 2A = 2 \sin A \cos A, \] (12)
\[ \cos 2A = \cos^2 A - \sin^2 A. \] (13)
6 Identities for sine squared and cosine squared

If we have $A = B$ in equation (10) then we find

$$\cos A \cos B = \frac{1}{2} \cos(A - A) + \frac{1}{2} \cos(A + A)$$

$$\cos^2 A = \frac{1}{2} \cos 0 + \frac{1}{2} \cos 2A.$$  

Simplifying this and doing the same with equation (11) we find the two identities

$$\cos^2 A = \frac{1}{2}(1 + \cos 2A), \quad (14)$$

$$\sin^2 A = \frac{1}{2}(1 - \cos 2A). \quad (15)$$

7 Identities involving tangent

Finally, from equations (2) and (3) we can obtain an identity for $\tan(A + B)$:

$$\tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}.$$  

Now divide numerator and denominator by $\cos A \cos B$ to obtain the identity we wanted:

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}. \quad (16)$$

We can get the identity for $\tan(A - B)$ by replacing $B$ in (16) by $-B$ and noting that tangent is an odd function:

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}. \quad (17)$$

8 Summary

There are many other identities that can be generated this way. In fact, the derivations above are not unique — many trigonometric identities can be obtained many different ways. The idea here is to be very familiar with a small number of identities so that you are comfortable manipulating and combining them to obtain whatever identity you need to.