Alternating Series

Recall that the Harmonic Series \( \sum_{k=1}^{\infty} \frac{1}{k} \) diverges even though the terms in the series approach zero.

In general, the fact that the terms in a series approach zero is not sufficient to guarantee that the series will converge. We do know, however, that if the terms do not approach zero then the series will not converge.

Consider \( \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots \). This is the Alternating Harmonic Series. The sequence of partial sums starts as

\[
\begin{align*}
S_1 &= 1 = 1 \\
S_2 &= 1 - \frac{1}{2} = \frac{1}{2} = 0.5 \\
S_3 &= \frac{1}{2} + \frac{1}{3} = \frac{5}{6} = 0.833 \\
S_4 &= \frac{5}{6} - \frac{1}{4} = \frac{7}{12} = 0.583 \\
S_5 &= \frac{7}{12} + \frac{1}{5} = \frac{47}{60} = 0.783 \\
S_6 &= \frac{47}{60} - \frac{1}{6} = \frac{37}{60} = 0.616 \\
S_7 &= \frac{37}{60} + \frac{1}{7} = \frac{297}{210} = 0.75952380
\end{align*}
\]

Notice that each \( S_{n+1} \) is between \( S_n \) and \( S_{n-1} \). This pattern suggests that the sequence will have a limit and so the series will converge.

\[
\text{Def. The series } \sum_{k=1}^{\infty} (-1)^{k+1} a_k \text{ with } a_k > 0 \text{ is called an alternating series.}
\]

If an alternating series has any hope of converging then \( \lim_{k \to \infty} a_k = 0 \) because of the kth term test.

Let \( S_n = \sum_{k=1}^{n} (-1)^{k+1} a_k \) be the nth partial sum.
Alternating Series

Consider the sequence of partial sums with even indexes \( \{S_{2n}\} \).

We want to show that this sequence is monotonic increasing and bounded if \( 0 < a_{k+1} \leq a_k \).

\[
S_{2n+2} - S_{2n} = \sum_{k=1}^{2n+2} (-1)^k a_k - \sum_{k=1}^{2n} (-1)^k a_k \\
= (-1)^{2n+3} a_{2n+2} + (-1)^{2n+2} a_{2n+1} \\
= a_{2n+1} - a_{2n+2} \geq 0 \quad \text{if} \quad a_{k+1} \leq a_k
\]

So \( S_{2n+2} - S_{2n} \geq 0 \Rightarrow S_{2n+2} \geq S_{2n} \) and so the sequence \( \{S_{2n}\} \) is monotonic increasing.

The sequence is bounded because

\[
o < a_1 + (-a_2 + a_3) + (-a_4 + a_5) + \ldots + (-a_{2n-2} + a_{2n-1}) - a_{2n} \leq a_1
\]

Since each term in parenthesis is negative.

\( \Rightarrow \{S_{2n}\} \), a bounded monotonic sequence, converges to some number \( L \).

The sequence \( \{S_{2n+1}\} \) also converges to \( L \) since

\[
\lim_{n \to \infty} S_{2n+1} = \lim_{n \to \infty} S_{2n} + a_{2n+1} = \lim_{n \to \infty} S_{2n} + \lim_{n \to \infty} a_{2n+1} = L + 0 = L
\]

Since \( \lim_{n \to \infty} S_{2n} = \lim_{n \to \infty} S_{2n+1} = L \) we can write \( \lim_{n \to \infty} S_n = L \) and so the sequence of partial sums converge to \( L \).

**Theorem Alternating Series Test:** Suppose that \( \lim_{k \to \infty} a_k = 0 \) and \( 0 < a_{k+1} \leq a_k \) for all \( k \geq 1 \). Then

\[
\sum_{k=1}^{\infty} (-1)^k a_k
\]

converges.
Alternating Series

To show that an alternating series converges using this test, you need to check three things:

1) $0 < a_k \quad k = 1, 2, \ldots$
2) $a_{k+1} \leq a_k \quad k = 1, 2, 3, \ldots$ i.e. $\frac{a_{k+1}}{a_k} \leq 1$
3) $\lim_{k \to \infty} a_k = 0$

Ex. Does $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\ln(k+1)}$ converge?

1) $\frac{1}{\ln(k+1)} > 0$ since $\ln(k+1) > 0$ for $k = 1, 2, 3, \ldots$

2) $\ln(k+1) < \ln(k+2)$ since $\ln x$ is strictly increasing

Therefore $\frac{1}{\ln(k+1)} > \frac{1}{\ln(k+2)}$

3) $\lim_{k \to \infty} \frac{1}{\ln(k+1)} = 0$ since $\ln(k+1) \to \infty$ as $k \to \infty$

\[\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\ln(k+1)} \text{ converges.}\]

We can also show the alternating series test graphically.

The partial sums will continue to bracket $L$ and move toward it, if the conditions of the alternating series test are met.
Alternating Series

From this diagram we also see that

\[ |S_n - L| \leq a_{n+1} \]

This gives us a useful tool to determine a bound for the error we make when approximating the sum of a convergent alternating series with a finite sum.

**Thm.** Suppose \( \sum_{k=1}^{\infty} (-1)^{k+1} a_k \) is a convergent alternating series that converges to \( S \). The error made by approximating \( S \) with \( S_n \) is less than \( a_{n+1} \):

\[ |S - S_n| \leq a_{n+1} \]

**Ex.** How many terms are necessary to compute \( \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \) to within \( 10^{-5} \) of the correct sum?

\[ a_k = \frac{1}{k} \]

So \( |S - S_n| \leq \frac{1}{n+1} \leq 10^{-5} \Rightarrow n+1 \geq 10^5 \Rightarrow n \geq 10^5 - 1 \)

Thus \( n \geq 10^5 - 1 \)

\[ \sum_{k=1}^{100001} \frac{(-1)^{k+1}}{k} \]

will approximate \( \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \) to within \( 10^{-5} \).

\[ \sum_{k=1}^{100001} \frac{(-1)^{k+1}}{k} = 0.69315218 \]

So the sum is approximately 0.69315.

Interestingly (and surprisingly)

\[ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \ln 2. \]