The Exponential Function Revisited

Back in Chapter 0 the irrational number \( e \) was introduced and defined as
\[
e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n
\]
but not much justification was given for this.

We have been using \( e \) [called "e" in honor of Euler] primarily as the base in the exponential function \( e^x \).

But what is this function really? How should we interpret this given that \( e \) is irrational and \( x \) can be irrational?

**DEF** \( e \) is the number for which \( \ln e = 1 \).

This definition uses the \( \ln x \) function previously defined as
\[
\ln x = \int_1^x \frac{1}{t} \, dt
\]
so we could also say that \( e \) is the number for which
\[
\int_1^e \frac{1}{t} \, dt = 1
\]

Starting from this definition we will explore the function \( e^x \) and find that it is the same function we have been using all along.

Question: What does \( 2^x \) mean?

"2 raised to the \( x \) power" – but what does this mean?
"2 times itself \( x \) times" – OK – if \( x \) is a whole number...

Perhaps we don’t understand this simple operation as much as we thought...

We proceed by examining \( e^x \).
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Case 1: $x$ is an integer.

If $x=0$ then $e^x = 1$.

If $x>0$ then $e^x = \underbrace{ee...e}_{x\text{ factors}}$.

If $x<0$ then let $y=-x$ so $e^x = \frac{1}{e^y} = \underbrace{eee...e}_{y\text{ factors}}$.

Case 2: $x$ is rational: $x = \frac{p}{q}$ for integers $p$ and $q$.

If $p \geq 0, q > 0$ then $e^x = e^{\frac{p}{q}} = (e^p)^{\frac{1}{q}} = q\sqrt[e^p]{e}$.

Case 1 covers $e^p$, and, although they may be difficult to find, we know in principle how to find $n^{th}$ roots of real numbers.

If $p \leq 0, q < 0$ then $e^x = e^{\frac{p}{q}} = e^{-\frac{p}{q}} = \frac{1}{q\sqrt[e^{-p}]{e}}$.

If $p \leq 0, q > 0$ or $p > 0, q < 0$ then $e^x = e^{\frac{p}{q}} = e^{-\frac{p}{q}}$.

So $e^x = \sqrt[q]{e^{-p}}$.

Case 3: $x$ is irrational. Now we proceed slowly.

Let $f(x) = \ln x$, with $x>0$. Observe that $f'(x) = \frac{1}{x} > 0$.

So $f$ is strictly increasing and therefore is one-to-one and invertible.

When $x$ is rational we notice that $\ln(e^x) = x \cdot \ln e$, a property already demonstrated. Since we defined $e$ as the number for which $\ln e = 1$ we have
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\[ \ln(e^x) = x \ln e = x \]

or

\[ f(e^x) = x \]

This shows that, at least for rational numbers \( x \),

\[ f^{-1}(x) = e^x \]

Since \( f^{-1}(x) = \ln x \) "undoes" what \( e^x \) "does".

Because we can find two rationals which bracket any given irrational within an arbitrarily small interval, we decide to define \( e^x \) for irrational \( x \) as:

**DEF** For irrational \( x \) we define \( y = e^x \) to be that number for which

\[ \ln y = \ln e^x = x \]

So, if \( x \) is irrational, \( e^x \) is defined to be the number \( y \) that satisfies \( \ln y = x \).

This means that \( \ln x = \log_e x \), i.e. \( \ln x \) is the base \( e \) logarithm.

So, we know how to compute \( e^x \) for all real numbers \( x \).

Finally, since \( e^x \) is the inverse of \( \ln x \), we have that \( \ln x \) is the inverse of \( e^x \):

\[ \ln e^x = x \quad , \quad e^{\ln x} = x \]
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Thm. For \( r, s \) any real numbers and \( t \) any rational number:

(i) \( e^{r+s} = e^r e^s \)

(ii) \( \frac{e^r}{e^s} = e^{r-s} \)

(iii) \( (e^r)^t = e^{rt} \)

Proof of i) \( \ln (e^{r+s}) = \ln (e^r) + \ln (e^s) = r \ln e + s \ln e = r + s \)

\( e^{\ln (e^{r+s})} = e^{r+s} = e^{r+s} \)

How are we to compute functions like \( 2^x \) when \( x \) is irrational?

If we want to compute \( a^x \) we can set \( y = a^x \).

Then \( y = e^{\ln y} = e^{\ln a^x} = e^{x \ln a} \)

We know how to compute \( e^x \) so we can compute \( e^{(x \ln a)} \)

\[ a^x = e^{(x \ln a)} \]

Derivative of \( e^x \)

What is \( \frac{d}{dx} e^x \)?

Start with \( y = e^x \); so \( x = \ln y \).

Differentiating implicitly:

\[ \frac{d}{dx} x = \frac{d}{dx} \ln y \quad \Rightarrow \quad 1 = \frac{1}{y} \cdot \frac{dy}{dx} \]
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So \[
\frac{dy}{dx} = y \Rightarrow \frac{d}{dx} e^x = e^x
\]

This also means \[
\int e^x \, dx = e^x + C
\]

How about \( \frac{d}{dx} a^x \)?

\[
\frac{d}{dx} a^x = \frac{d}{dx} e^{\ln a^x} = \frac{d}{dx} e^{\ln a} = a^x \ln a
\]

\[
\int a^x \, dx = \frac{a^x}{\ln a} + C
\]

Examples

Find the extrema and inflection points of \( f(x) = xe^{2x+1} \)

\[
f'(x) = e^{2x+1} + 2xe^{2x+1}
\]

Continuous so extrema are found where \( f'(x) = 0 \)

\[
e^{2x+1} + 2xe^{2x+1} = 0
\]

\[
1 + 2x = 0
\]

\[
x = -\frac{1}{2}
\]

\[
f''(x) = 2e^{2x+1} + 2e^{2x+1} + 4xe^{2x+1} = 4(1+x)e^{2x+1}
\]

Since \( f''(-\frac{1}{2}) = 2 > 0 \) we know \( f(x) \) is concave up at \( x = -\frac{1}{2} \)

so \( x = -\frac{1}{2} \) is the location of a minimum

Local minimum of \( f(x) = xe^{2x+1} \) occurs at \( x = -\frac{1}{2} \)
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\[ f''(x) = 0 \quad \text{gives} \quad 0 = 4(1 + x)e^{2x+1} \]
\[ 0 = 1 + x \]
\[ x = -1 \]

When \( x < -1 \), \( f''(x) < 0 \) so curve is concave down.
When \( x > -1 \), \( f''(x) > 0 \) so curve is concave up.

\[ x = -1 \] is an inflection point of \( f(x) = xe^{2x+1} \)

**Ex.** \( \int 3^{2x} \, dx \)
\[ u = 2x \quad du = 2 \, dx \]
\[ \frac{1}{2} \int 3^u \, du = \frac{1}{2} \frac{3^u}{\ln 3} + C \]

\[ = \frac{3^{2x}}{2\ln 3} + C \]

**Ex.** \( \int \frac{2^{\ln x}}{x} \, dx \)
\[ u = \ln x \quad du = \frac{1}{x} \, dx \]
\[ \int 2^u \, du = \frac{2^u}{\ln 2} + C = \frac{2^{\ln x}}{\ln 2} + C \]

**Ex.** \( \int \sin x \, e^{1 + \cos x} \, dx \)
\[ u = 1 + \cos x \quad du = -\sin x \, dx \]
\[ -\int -\sin x \, e^{1 + \cos x} \, dx = -\int e^u \, du = -e^u + C = -e^{1 + \cos x} + C \]

**OR.** 
\[ \int \sin x \, e^{1 + \cos x} \, dx = \int \sin x \, e^u \, du \]
\[ = -e^u + C \quad \text{if} \ u = \cos x \]
\[ = -e[e^u + C] = -e e^{\cos x} - ce = -e^{1 + \cos x} - ce \]

just a constant...