Infinite Series

Atenolol is a drug frequently prescribed for hypertension, and a typical dose is 50 mg per day.

24 hours after a dose, approximately 7% of the Atenolol remains in the body, the other 93% having been removed.

<table>
<thead>
<tr>
<th>Day</th>
<th>Amount in body (mg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>50</td>
</tr>
<tr>
<td>1</td>
<td>50 + 0.07(50)</td>
</tr>
<tr>
<td>2</td>
<td>50 + 0.07[50 + 0.07(50)]</td>
</tr>
<tr>
<td>3</td>
<td>50 + 0.07[50 + 0.07[50 + 0.07(50)]]</td>
</tr>
</tbody>
</table>

From the pattern we see, we can determine that the amount of Atenolol in the body immediately after the dose on the n\textsuperscript{th} day is

\[ Q_n = 50 + 50(0.07) + 50(0.07)^2 + \ldots + 50(0.07)^n \]

or

\[ Q_n = \sum_{k=0}^{n} 50(0.07)^k \]

Suppose we want to know how much Atenolol is present in the body after 1 year of daily doses, after 10 years, or after 50 years. We might even wonder what

\[ \sum_{k=0}^{\infty} 50(0.07)^k \]

works out to be.

This last sum is an Infinite Series and is the sum of the terms in a sequence.

Actually, this is a special type of series called a geometric series. The general geometric series is

\[ a + ar + ar^2 + ar^3 + \ldots + ar^k + \ldots \]
Consider the following, and see if there is any slight-of-hand.

Let $L_n = a + ar + ar^2 + ... + ar^{n-1} + ar^n$

Then $rL_n = ar + ar^2 + ar^3 + ... + ar^{n} + ar^{n+1}$

Both of these sums are finite and so clearly exist. Next, form

$L_n - rL_n = a + ar + ar^2 + ... + ar^{n-1} + ar^n - ar^2 - ... - ar^{n-1} - ar^2 - ar^{n+1}$

We see that this sum simplifies dramatically since the $ar$ terms cancel, the $ar^2$ terms cancel, etc. We say the sum telescopes. All that remains is

$L_n - rL_n = a - ar^{n+1}$

We can solve this for $L_n$ to get

$L_n = \frac{a(1-r^{n+1})}{1-r}$

The sum of $a + ar + ar^2 + ... + ar^n$ is given by

$$\sum_{k=0}^{n} ar^k = \frac{a(1-r^{n+1})}{1-r}$$

Now - what happens if $n \to \infty$? What is $\sum_{k=0}^{\infty} ar^k$?

From the above, it makes sense that to compute the infinite sum we need

$$\lim_{n \to \infty} L_n = \lim_{n \to \infty} \frac{a(1-r^{n+1})}{1-r}$$

Note that $\lim_{n \to \infty} r^{n+1}$ converges to zero if $r < 1$ but diverges if $r \geq 1$. 

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This means that \( \lim_{n \to \infty} a(1 - r^n) = \frac{a}{1-r} \) if \( r < 1 \).

The geometric series \( \sum_{k=0}^{\infty} ar^k \) converges to \( \frac{a}{1-r} \)
if \( |r| < 1 \) and diverges if \( |r| \geq 1 \).

In our starting example, \( a = 50 \) and \( r = 0.07 < 1 \), so after an infinite number of days the amount of Atenolol in the body is \( \frac{50}{1-0.07} \text{ mg} = 537.634 \text{ mg} \).

Hm... is this possible? Can we really add an infinite number of terms and get a finite value?

This is one of the big questions in the study of infinite series.

\[
\text{Def } \sum_{k=k_0}^{\infty} a_k \text{ is an infinite series; it is the sum of the terms in the sequence } \{a_n\}_{n=k_0}^{\infty}.
\]

Let \( S_n = \sum_{k=k_0}^{n} a_k \) be the sum of the terms up to \( a_n \). Since this sum is finite, we know that we can compute it. \( S_n \) is a partial sum of the series.

\[
\text{Def } \text{The series } \sum_{k=k_0}^{\infty} a_k \text{ converges to } S \text{ if the sequence of partial sums } \{S_n\}_{n=k_0}^{\infty} \text{ converges to } S. \text{ The series diverges if the sequence of partial sums diverges.}
\]
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Ex \[ \sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots = \frac{1}{2} (1 + \frac{1}{2} + (\frac{1}{2})^2 + (\frac{1}{2})^3 + \ldots) \]

This is a geometric series with: \( a = \frac{1}{2} \) and \( r = \frac{1}{2} \).

Since \( r = \frac{1}{2} < 1 \) we know that the series will sum to \( \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1 \Rightarrow \sum_{k=1}^{\infty} \frac{1}{2^k} = 1 \)

Ex \[ \sum_{k=1}^{\infty} 2^k = 2 + 2^2 + 2^3 + \ldots = 2 (1 + 2 + 2^2 + 2^3 + \ldots) \]

Geometric series with \( a = 2 \), \( r = 2 \).

Since \( r = 2 > 1 \) this series diverges.

Ex \[ \sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)} \] Not a geometric series.

Try partial fractions: \[ \frac{1}{(2k-1)(2k+1)} = \frac{A}{2k-1} + \frac{B}{2k+1} \]

\[ A = \frac{1}{2}, \quad B = -\frac{1}{2} \]

\[ S_n = \sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{1}{2} \sum_{k=1}^{n} \frac{1}{2k-1} - \frac{1}{2} \sum_{k=1}^{n} \frac{1}{2k+1} \]

(We know we can do this since the sums are finite)

\[ \therefore S_n = \frac{1}{2} \left[ \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1} \right] - \frac{1}{2} \left[ \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots + \frac{1}{2n-1} + \frac{1}{2n+1} \right] \]

\[ = \frac{1}{2} \left[ 1 + \left( \frac{1}{3} - \frac{1}{3} \right) + \left( \frac{1}{5} - \frac{1}{5} \right) + \left( \frac{1}{7} - \frac{1}{7} \right) + \ldots + \left( \frac{1}{2n-1} - \frac{1}{2n-1} \right) - \frac{1}{2n+1} \right] \]

This sum telescopes to \( S_n = \frac{1}{2} [1 - \frac{1}{2n+1}] \)

Since \( \lim_{n \to \infty} \frac{1}{2} [1 - \frac{1}{2n+1}] = \frac{1}{2} (1-0) = \frac{1}{2} \)

we know that our series converges and \[ \sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)} = \frac{1}{2} \]
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Unfortunately, we will not be able to find the sum of most convergent series, and we will need to content ourselves with determining whether or not the series converges.

We begin by noting that if the terms in the series do not tend to zero then the series cannot converge.

\[ \text{Thm: } K^{th} \text{ term test} \]

If \( \lim_{k \to \infty} a_k \neq 0 \) then the series \( \sum_{k=k_0}^{\infty} a_k \) diverges.

\[ \text{Note: The converse is not (in general) true.} \]

Ex. We will soon see that the Harmonic Series \( \sum_{k=1}^{\infty} \frac{1}{k} \) diverges even though the terms approach zero.

Ex. \( \sum_{k=1}^{\infty} \frac{1}{k+1} \) must diverge since \( \lim_{k \to \infty} \frac{1}{k+1} = 1 \neq 0 \).

\[ \text{Thm} \]

1) If \( \sum_{k=1}^{\infty} a_k \) converges to \( A \) and \( \sum_{k=1}^{\infty} b_k \) converges to \( B \)

then \( \sum_{k=1}^{\infty} (a_k \pm b_k) \) converges to \( A \pm B \) and

\( \sum_{k=1}^{\infty} c a_k = c A \) for any constant \( c \).

2) If \( \sum_{k=1}^{\infty} a_k \) converges but \( \sum_{k=1}^{\infty} b_k \) diverges then

\( \sum_{k=1}^{\infty} (a_k \pm b_k) \) diverges.