Sequences of Real Numbers

Consider the function \( f(x) = \frac{1}{x} \).

What is the domain of this function?

What is the range?

Chances are the domain will be given as all reals except zero and the range will also be all reals except zero.

Suppose, however, the domain is a set of integers greater than or equal to some integer \( n_0 \). In this case the range of \( f(n) \) includes rational numbers.

**Def**: A sequence is a list of numbers generated by a function whose domain is the set of integers greater than some integer \( n_0 \).

We will use \( a_n \) to refer to the \( n^{th} \) term of a sequence and will often specify the function that generates the sequence as

\[ a_n = f(n). \]

**Ex.** If \( a_n = \frac{1}{n} \) then the sequence is \( 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots, \frac{1}{n}, \ldots \)

**Ex.** If \( a_n = \frac{1}{\sqrt{n}} \) then the sequence is \( 1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{4}}, \frac{1}{\sqrt{5}}, \ldots \)

**Ex.** If \( a_n = \sin\left(\frac{n\pi}{2}\right) \) then the sequence is \( 1, 0, -1, 0, 1, 0, -1, 0, \ldots \)

In each of the above examples, \( n_0 = 1 \). We do not need to start with \( 1 \), but we will always start with some integer value for the index \( n \), and then continue as the index goes to infinity.

**Ex.** The sequence given by \( a_n = \frac{n+1}{n}, \; n_0 = 5 \), is

\[ \frac{6}{5}, \frac{7}{6}, \frac{8}{7}, \frac{9}{8}, \frac{10}{9}, \frac{11}{10}, \ldots \]
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From now on we will use $\{a_n\}$ to denote a sequence. If $a_n = \frac{1}{n}$ then
\[
\{a_n\}_{n=1}^{\infty} = \left\{ \frac{1}{n} \right\}_{n=1}^{\infty} = \{1, \frac{1}{2}, \frac{1}{3}, \ldots \}
\]

**Def.** The sequence $\{a_n\}_{n=m}^{\infty}$ converges to $L$ if and only if given any number $\varepsilon > 0$, there is an integer $N$ for which
\[
|a_n - L| < \varepsilon \quad \text{for every } n > N.
\]
If there is no such number $L$ then the sequence diverges.

**Ex.** Show that $\{\frac{1}{\sqrt{n}}\}_{n=1}^{\infty}$ converges to zero.

From the definition, we need to show that we can make
\[
|\frac{1}{\sqrt{n}} - 0| = \frac{1}{\sqrt{n}}
\]
as small as we want by finding $N$ such that $n > N$.

Require $\frac{1}{\sqrt{n}} - 0 < \varepsilon \Rightarrow \frac{1}{\sqrt{n}} < \varepsilon$

\[
\frac{1}{\varepsilon^2} < n
\]

So, if $N$ is an integer s.t. $N \geq \frac{1}{\varepsilon^2}$, we are sure that $|\frac{1}{\sqrt{n}}| < \varepsilon$.

**General Approach:** Require $|a_n - L| < \varepsilon$ and solve for $n$ in terms of $\varepsilon$. 

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Theorem: Suppose \( \{a_n\}_{n=n_0}^\infty \) and \( \{b_n\}_{n=n_0}^\infty \) both converge. Then

\[
egin{align*}
1) \lim_{n \to \infty} (a_n + b_n) &= \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n \\
2) \lim_{n \to \infty} (a_n - b_n) &= \lim_{n \to \infty} a_n - \lim_{n \to \infty} b_n \\
3) \lim_{n \to \infty} (a_n \cdot b_n) &= (\lim_{n \to \infty} a_n)(\lim_{n \to \infty} b_n) \\
4) \lim_{n \to \infty} \frac{a_n}{b_n} &= \frac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n} \text{ if } \lim_{n \to \infty} b_n \neq 0.
\end{align*}
\]

Note: This theorem only applies to convergent sequences. We cannot use it to say

\[
\lim_{n \to \infty} \frac{n}{n^2} = 0 = \frac{\lim_{n \to \infty} n}{\lim_{n \to \infty} n^2} \text{ since neither of these last two series converge.}
\]

Ex. \[
\lim_{n \to \infty} \frac{3n+1}{2n+5} = \lim_{n \to \infty} \frac{3 + \frac{1}{n}}{2 + \frac{5}{n}} = \frac{3+0}{2+0} = \frac{3}{2} \quad \text{Converges}
\]

We cannot use L'Hôpital's rule here, since the function is defined for integers and not continuous and not differentiable.

Ex. \[
\lim_{n \to \infty} \frac{n^2+n}{2+3n} = \lim_{n \to \infty} \frac{1 + \frac{1}{n}}{\frac{2}{n} + \frac{3}{n}} \to \frac{1}{0} = \infty
\]

So the sequence \( \{ \frac{n^2+n}{2+3n} \}_{n=n_0}^\infty \) diverges for all choices of \( n_0 \).

Ex. Consider \( \{(-1)^n\}_{n=1}^\infty \). Does this converge?

The sequence is \( \{-1, 1, -1, 1, \ldots\} \). Clearly the terms in this sequence do not approach a fixed value \( L \), so the sequence diverges.

* Terms do not need to go to infinity *
Sequences of Real Numbers

We can, in some cases, use l'Hôpital's rule.

**Theorem:** If \( \lim_{x \to \infty} f(x) = L \) then \( \lim_{n \to \infty} f(n) = L \) also.

**Ex.** Does \( \frac{\ln n}{n} \) converge?

Let \( f(x) = \frac{\ln x}{x} \). As \( x \to \infty \) this function has the indeterminate form \( \frac{\infty}{\infty} \). Using l'Hôpital's rule we have

\[
\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = \lim_{x \to \infty} \frac{1}{x} = 0
\]

So,

\[
\lim_{n \to \infty} \frac{\ln n}{n} = 0
\]

and the sequence converges to 0.

**Theorem:** (Squeeze or Sandwich Theorem)

Suppose \( \{a_n\} \to a_0 \) and \( \{b_n\} \to a_0 \) both converge to \( L \).

If there is an integer \( n_0 \) such that for all \( n \geq n_0 \), we have \( a_n \leq c_n \leq b_n \), then \( \{c_n\} \) also converges to \( L \).

**Ex.** \( \left\{ \frac{\cos^2(n)}{n} \right\}_{n=1}^{\infty} \)

Notice that \( \frac{0}{\infty} \) and \( \frac{1}{n} \) both converge to zero.

\[ 0 \leq \frac{\cos^2 n}{n} \leq \frac{1}{n} \]

Since \( 0 \leq \cos^2 n \leq 1 \).

As the sequences \( \{0\} \) and \( \{\frac{1}{n}\} \) converge to zero, they keep the terms in \( \left\{ \frac{\cos^2 n}{n} \right\} \) sandwiched between them, so they "squeeze" this sequence down to zero.
Ex. Prove that if \( \{|a_n|\}_{n=1}^{\infty} \) converges to zero then \( \{a_n\}_{n=1}^{\infty} \) converges to zero as well.

If \( \{a_n\}_{n=1}^{\infty} \) converges to zero then so does \( \{-a_n\}_{n=1}^{\infty} \).

(how do we know? Part 3 of Thm on page 3.)

Since \( -|a_n| \leq a_n \leq |a_n| \), we know by the squeeze thm that

\[ \{a_n\}_{n=1}^{\infty} \]

will converge to zero as well.

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Def. The sequence \( \{a_n\}_{n=1}^{\infty} \) is **increasing** (or nondecreasing) if

\[ a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots \]

or is **decreasing** (or nonincreasing) if

\[ a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots \]

A sequence is monotonic increasing if it is nondecreasing and monotonic decreasing if it is nonincreasing.

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Def. The sequence \( \{a_n\}_{n=1}^{\infty} \) is **bounded** if there is a number \( M > 0 \) for which \( |a_n| \leq M \) for all \( n \).

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Ex. \( \{1 + \frac{1}{n}\}_{n=1}^{\infty} \) is monotonic decreasing since \( 1 + \frac{1}{n} > 1 + \frac{1}{(n+1)} \) for all \( n > 1 \). This is true because \( n < n+1 \), so \( \frac{1}{n} > \frac{1}{n+1} \), \( \Rightarrow 1 + \frac{1}{n} > 1 + \frac{1}{(n+1)} \).

\[ \{1 + \frac{1}{n}\}_{n=1}^{\infty} \] is bounded by 2 since

\[ n \geq 1 \]

\[ \Rightarrow 0 \leq \frac{1}{n} \leq 1 \]

\[ \Rightarrow 0 \leq 1 + \frac{1}{n} \leq 2 \]

\[ \Rightarrow |1 + \frac{1}{n}| \leq 2 \ \text{for} \ n \geq 1 \]
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Thm: Every bounded monotonic sequence converges.

Ex: Does \( \frac{2^n}{n!} \) converge?

Let us try to determine if the sequence is monotonic.

We can do this by examining how the terms in the sequence change.

In particular, if the ratio \( \frac{a_{n+1}}{a_n} \) is less than one as \( n \to \infty \), then the sequence is decreasing; and if it is greater than one then the sequence is increasing.

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{2^{n+1}}{2^n \cdot n!} = \lim_{n \to \infty} \frac{2^{n+1}}{2^n \cdot (n+1) \cdot n!} = \lim_{n \to \infty} \frac{2 \cdot 2^n \cdot n!}{2^n \cdot (n+1) \cdot n!} = \lim_{n \to \infty} \frac{2}{n+1} = 0
\]

From this we conclude that the sequence is monotonic decreasing.

To see that the sequence is bounded, notice that

\[
2 \geq \frac{2}{1} > \frac{2 \cdot 2}{1 \cdot 2} > \frac{2 \cdot 2 \cdot 2}{1 \cdot 2 \cdot 3} > \ldots > \frac{2^n}{n!} > 0
\]

So the sequence is bounded between 2 and \( \infty \).

By the theorem above, since the sequence is monotonic and bounded, it must converge.

The Completeness Axiom

If a non-empty set \( S \) of real numbers has a lower bound, then it has a greatest lower bound. Equivalently, if it has an upper bound, it has a least upper bound.