Volume by Slicing

What formulas for volume can you think of?

Rectangular box: \(V = lwh\)
Sphere: \(V = \frac{4}{3}\pi r^3\)
Cylinder: \(V = \pi r^2h\)
Cone: \(V = \frac{1}{3}\pi r^2h\)

Where do these formulas come from?

What is the volume of this solid?

The area of the triangle at the end is \(A = \frac{1}{2}(12\text{ in})(5\text{ in}) = 30\text{ in}^2\). To find the volume we need to multiply by 35 in (the length) to get \(V = 1050\text{ in}^3\).

Notice that we could also do this by "slicing" along the long axis to get \(n\) slices each with thickness \(dx\).

The volume of this slice in \(\Delta V = A\Delta x\).

The volume of the solid is \(V = \sum_{i=1}^{n} A\Delta x\),

where \(\Delta x = \frac{35}{n}\text{ in}\) and \(A = 30\text{ in}^2\).
Volume by Slicing

We can use this approach when the cross-sectional area changes.

Example: Find the volume of a right circular cone with height $h$ and base radius $r$.

\[ \text{Find the little piece of volume for each slice} \]

\[ \Delta V = \pi x^2 \Delta y \]

Notice that $x$ will vary depending on which slice we consider.

\[ \frac{\text{Express } \Delta V \text{ in terms of the variable corresponding to the direction perpendicular to the slices,}}{} \]

The in the "$\Delta$" direction - in this case $\Delta y$

So we need the $y$ variable.

In this case we need $x$ in terms of $y$ - we will use similar triangles

\[ \frac{x}{y} = \frac{r}{h} \]

so

\[ x = \frac{r}{h} y \]

\[ \therefore \Delta V = \pi \left( \frac{r}{h} y \right)^2 \Delta y = \frac{\pi r^2}{h^2} y^2 \Delta y \]
Volume by Slicing

3. Now we just need to add up the little pieces! Since we want the exact volume we will use an infinite sum.

\[ V = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{\pi r^2}{h^2} y_i^2 \Delta y \]

\[ = \int_{0}^{h} \frac{\pi r^2}{h^2} y^2 \, dy \quad \text{Note the limits on } y = \text{the bounds of our slices} \]

\[ V = \frac{\pi r^2}{h^2} \left[ \frac{y^3}{3} \right]_{0}^{h} = \frac{\pi r^2 h^3}{3 h^2} - 0 = \frac{1}{3} \pi r^2 h \]

Which is the formula for the volume of this cone.

Ex. Find the volume of the solid whose base bounded by the parabola \( y = x^2 \) between \( y = 0 \) and \( y = 4 \) and whose cross sections perpendicular to the y axis are squares.

1. \( \Delta V = (2x)^2 \Delta y = 4x^2 \Delta y \). Now need x in terms of y.

2. Use \( y = x^2 \) so \( \Delta V = 4y \Delta y \)

3. \( V = \int_{0}^{4} 4y \, dy = 2y^2 \bigg|_{0}^{4} = 2 \cdot 16 - 0 = 32 \text{ Square units} \)
Volume by Slicing

Ex. Find the volume of the wedge of a cylinder shown here:

A) Slices are triangles

\[ dv = \frac{1}{2} \cdot y \cdot h \, dx \quad y^2 + x^2 = l^2 \quad y = l - x^2 \]

\[ h = y \]

So \[ dv = \frac{1}{2} y^2 \, dx = \frac{1}{2} (l-x^2) \, dx \]

\[ V = 2 \int_0^1 \frac{1}{2} (l-x^2) \, dx \]

by symmetry

\[ V = \int_0^1 1 - x^2 \, dx = \left[ x - \frac{x^3}{3} \right]_0^1 = 1 - \frac{1}{3} = \frac{2}{3} \]

B) Slices are rectangles \quad width = 2x

\[ x^2 + y^2 = l^2 \Rightarrow x^2 = l^2 - y^2 \]

\[ \begin{align*}
  dv &= 2x \cdot h \, dy = 2\sqrt{1-y^2} \cdot y \, dy = 2y\sqrt{1-y^2} \, dy \\
  V &= \int_0^1 2y\sqrt{1-y^2} \, dy \\
  &\quad \text{Let} \quad u = 1-y^2 \quad du = -2y \, dy \\
  &\quad u(0) = 1 \quad u(1) = 0 \\
  V &= -\int_0^1 \sqrt{u} \, du \\
  &= \int_0^1 u^{\frac{1}{2}} \, du = \frac{2}{3} u^{\frac{3}{2}} \bigg|_0^1 = \frac{2}{3}
\end{align*} \]