Volume of Revolution

Ex. Find the volume that results when the area below the curve \( y = 5 - \frac{1}{2}x \), \( 0 \leq x \leq 8 \) is rotated about the \( x \)-axis.

We could look up the formula for the volume of the frustum of a cone, but why, when we can use calculus?

If we divide the area up into thin strips (here we go again!) and rotate each strip, we will generate a sequence of "disks". If the width of the strip is \( \Delta x \) and the height of the strip is \( f(x) \), then the volume of the disk will be

\[
\Delta V = \pi r^2 \Delta x = \pi (f(x))^2 \Delta x
\]

We can now use a Riemann sum to compute the approximate volume

\[
V \approx \sum_{i=1}^{n} \pi (f(x_i))^2 \Delta x, \quad \Delta x = \frac{b-a}{n}
\]
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As \( n \to \infty \) the sum becomes an integral and

\[
V = \int_{c}^{b} \pi (f(x))^2 \, dx
\]

So, for our problem

\[
dV = \text{"little piece of volume"}
\]

\[= \text{"disk"}
\]

\[
= \frac{\pi (5 - \frac{x}{2})^2 \, dx}{\text{area \over \text{thickness}}}
\]

\[
= \pi (25 - 5x + \frac{x^2}{4}) \, dx
\]

So

\[
V = \int_{0}^{8} \pi (25 - 5x + \frac{x^2}{4}) \, dx
\]

\[
= \pi \left[ 25x - \frac{5}{2}x^2 + \frac{x^3}{12} \right]_0^8
\]

\[
= \pi \left[ 200 - 160 + \frac{512}{3} \right]
\]

\[
= \frac{248\pi}{3}
\]

\[
\approx 259.705
\]

\[
V = 259.705 \text{ cubic units}
\]

Ex: Find the volume when \( y = x^2 \), \( 1 \leq x \leq 2 \), is rotated about the \( x \)-axis.

\[
dV = \pi r^2 \, dx, \quad r = y = x^2
\]

\[
= \pi (x^2)^2 \, dx
\]

\[
= \pi x^4 \, dx
\]

\[
V = \int_{1}^{2} \pi x^4 \, dx = \frac{\pi x^5}{5} \bigg|_1^2 = \pi \left[ \frac{32}{5} - \frac{1}{5} \right] = \frac{31\pi}{5}
\]

\[
V = \frac{31\pi}{5} \approx 19.4779 \text{ cubic units}
\]
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Sometimes we want to rotate about the y-axis.

Ex. Find the volume of the paraboloid formed when \( y = x^2 \) is rotated around the y-axis between \( y = 0 \) and \( y = 5 \).

The same as before, except now the radius of each disk is perpendicular to the y-axis.

\[
dV = \pi r^2 \, dy = \pi x^2 \, dy = \pi y \, dy \quad \text{since} \quad x^2 = y
\]

\[
V = \int_0^5 \pi y \, dy = \pi \left[ \frac{y^2}{2} \right]_0^5 = \frac{25\pi}{2} \approx 39.270 \quad \text{cubic units}
\]

Notice that our limits were in terms of \( y \).

Ex. What volume results when the area bounded by the x-axis, the curve \( y = \sqrt{x} \) and the line \( y = 2 - x \) is rotated about

a) the x axis,
b) the y axis,
c) the line \( y = 2 \)?
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First we graph the area.

a) Rotate about x-axis

Need two intervals

\[ dv_1 = \pi r^2 dx = \pi y^2 dx = \pi x^2 dx \text{ on } [0, 1] \]
\[ dv_2 = \pi r^2 dx = \pi y^2 dx = \pi (2-x)^2 dx \text{ on } [1, 2] \]

\[ V = \int_0^1 \pi x^2 dx + \int_1^2 \pi (4-4x+x^2) dx = \pi \left[ \frac{x^2}{2} \right]_0^1 + \pi \left[ 4x - 2x^2 + \frac{x^3}{3} \right]_1^2 \]

\[ = \pi \left[ \frac{1}{2} + (8 - 8 + \frac{8}{3}) - (4 - 2 + \frac{8}{3}) \right] = \pi \left[ \frac{1}{2} + \frac{8}{3} - 2 - \frac{8}{3} \right] \]

\[ = \pi \left[ \frac{2}{6} + \frac{16}{6} - \frac{12}{6} - \frac{2}{6} \right] = \pi \left[ \frac{5}{6} \right] = 2.618 \text{ cubic units} \]

b) Rotate about y-axis

This will give "washers" (disks with smaller disks missing from the center).

\[ dv = \pi r_1^2 dy - \pi r_2^2 dy = \pi (r_1^2 - r_2^2) dy \]
\[ = \pi (x_1^2 - x_2^2) dy \]

\[ y = 2 - x_1 \text{ so } x_1 = 2 - y \]
\[ y = \sqrt{x_2} \text{ so } x_2 = y^2 \]

\[ \therefore \quad dv = \pi \left[ (2-y)^2 - y^4 \right] dy \]
\[ = \pi \left[ 4 - 4y + y^2 - y^4 \right] dy \]
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\[ V = \int_{0}^{1} \pi (4 - 4y + y^2 - y^4) \, dy \]

\[ = \pi \left[ 4y - 2y^2 + \frac{y^3}{3} - \frac{y^5}{5} \right]_{0}^{1} \]

\[ = \pi \left[ 4 - 2 + \frac{1}{3} - \frac{1}{5} \right] \]

\[ = \pi \left[ \frac{32}{15} - \frac{2}{15} \right] = \frac{32\pi}{15} \approx 6.7021 \text{ cubic units} \]
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c) Rotate about the line $y = 2$.

We need washers once again, and we'll need to handle $0 \leq x \leq 1$ and $1 \leq x \leq 2$ separately.

On $0 \leq x \leq 1$ \[ dV = \pi (x)^2 \, dx - \pi (2 - y)^2 \, dx \quad \text{where} \quad y = \sqrt{x} \]

\[ = [4\pi - \pi (4 - 4\sqrt{x} + x)] \, dx \]

\[ = (4\pi \sqrt{x} - \pi x) \, dx \]

\[ V_1 = \int_0^1 4\pi \sqrt{x} - \pi x \, dx = 4\pi \int_0^1 \frac{2}{3} x^{3/2} - \pi \frac{x}{2} \, dx \left| \begin{array}{c} 1 \\ 0 \end{array} \right. = \frac{8\pi}{3} - \pi = \frac{13\pi}{6} \]

On $1 \leq x \leq 2$ \[ dV = \pi (x)^2 \, dx - \pi (2 - y)^2 \, dx \quad \text{where} \quad y = 2 - x \]

\[ = 4\pi x \, dx - \pi (2 - (2 - x))^2 \, dx \]

\[ = (4\pi - \pi x^2) \, dx \]

\[ V_2 = \int_1^2 4\pi - \pi x^2 \, dx = 4\pi x - \pi \frac{x^3}{3} \bigg| \begin{array}{c} 2 \\ 1 \end{array} = \frac{8\pi}{3} - \frac{8\pi}{3} - 4\pi + \frac{\pi}{3} \]

\[ = 4\pi - \frac{7\pi}{3} = \frac{5\pi}{3} \]

\[ V = \frac{13\pi}{6} + \frac{5\pi}{3} = \frac{13\pi}{6} + 10\pi = \frac{23\pi}{6} \approx 12.043 \text{ cubic units} \]