Volume of Revolution using Cylindrical Shells

Find the volume swept out when the area between the curve $y = 1 - x^2$ and the x-axis is rotated about the line $x = 3$.

![Graph of y = 1 - x^2](image)

We can proceed as we have before:

$$dv = \pi r_1^2 \, dy - \pi r_2^2 \, dy$$

where $r_1 = 3 + \sqrt{1-y}$ and $r_2 = 3 - \sqrt{1-y}$

So

$$dv = \pi ((3 + \sqrt{1-y})^2 - (3 - \sqrt{1-y})^2) \, dy$$

$$= \pi [(9 + 6\sqrt{1-y} + 1-y) - (9 - 6\sqrt{1-y} + 1-y)] \, dy$$

$$= \pi (12\sqrt{1-y}) \, dy$$

So

$$V = \int_0^1 12\pi \sqrt{1-y} \, dy$$

$u = 1-y$, $u(0) = 1$, $u(1) = 0$

$$- 12\pi \left[ \sqrt{u} \, du \right]_0^1 = 12\pi \frac{2}{3} u^{3/2} \bigg|_0^1 = 8\pi$$
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Alternatively, we could slice the area vertically and revolve them to form cylindrical shells.

If we cut the shell and unroll it, we will get a rectangular solid with thickness $dx$ (or $\Delta x$), height $y = 1 - x^2$, and length $2\pi r$ where $r$ is the radius of revolution.

In this case $r = 3 - x$.

So $dV = 2\pi r \cdot h \cdot dx$

$= 2\pi (3-x) \cdot (1-x^2) \, dx \Rightarrow V = \int_{-1}^{1} 2\pi (3-x)(1-x^2) \, dx$

$V = 2\pi \int_{-1}^{1} (3-x-3x^2+x^3) \, dx = 2\pi \left[ 3x - \frac{x^2}{2} - x^3 + \frac{x^4}{4} \right]_{-1}^{1}$

$V = 2\pi \left[ 3 - \frac{1}{2} - 1 + \frac{1}{4} \right] - 2\pi \left[ -3 - \frac{1}{2} + 1 + \frac{1}{4} \right]$

$V = 2\pi \left[ 6 - 2 \right] = 8\pi$

In this case, the integral was slightly more involved than in the previous approach, but finding $dV$ was more natural and less complicated.