We "solve" a differential equation such as \( \frac{dy}{dt} = f(t, y) \) by finding a function \( y(t) \) that satisfies it.

Example: \( \frac{dy}{dt} = 2y \)

We want to rewrite this so all "y"s appear on one side of an equation and all "t"s appear on the other.

\[
\frac{dy}{dt} = 2y \\
\frac{1}{2y} \frac{dy}{dt} = 1 \\
\frac{1}{2y} \, dy = dt
\]

Now integrate:

\[
\int \frac{1}{2y} \, dy = \int dt \\
\frac{1}{2} \ln|y| = t + C
\]

\[
\frac{1}{2} \ln|y| = t + C
\]

Now solve for \( y \):

\[
\ln|y| = 2t + 2C \\
|y| = e^{2t+2C} \\
|y| = e^{2t}e^{2C}
\]

What will we do with the absolute value signs? Note that \( e^{2C} \) is a positive constant. If we replace it with a new constant \( C \) that can be \( t \) we have →
\[ y = Ce^{2t} \]

This is a solution — actually it represents a family of solutions of \( \frac{dy}{dt} = 2y \) because \( C \) can be any constant (even zero).

Check: \( y' = 2Ce^{2t} = 2y \checkmark \)

Suppose, along with the differential equation we also have an initial condition \( y(0) = 1 \). This will uniquely determine the value of \( C \) and hence the function \( y \).

\[ y(0) = 1 = Ce^{2(0)} = Ce^0 = C \]

so \( C = 1 \).

The pair \( \begin{cases} \frac{dy}{dt} = 2y \\ y(0) = 1 \end{cases} \) is called an initial value problem or IVP.

Ex \( \frac{dy}{dx} = 3y - 2 \)

\[ \frac{1}{3y-2} \, dy = dx \]

\[ \int \frac{1}{3y-2} \, dy = \int dx \]

\[ \frac{1}{3} \ln |3y-2| = x + C \]

(chain rule)
\[ \frac{1}{3} \ln |3 y - 2| = x + c \]

\[ \ln |3 y - 2| = 3x + 3c \]

\[ |3 y - 2| = e^{3x} e^{3c} \]

\[ 3 y - 2 = C e^{3x} \]

\[ 3 y = C e^{3x} + 2 \]

\[ y = \frac{C}{3} e^{3x} + \frac{2}{3} \]

or, since \( \frac{C}{3} \) is yet another constant.

\[ y = C e^{3x} + \frac{2}{3} \]

Check \[ y' = 3C e^{3x} \]

\[ 3 y - 2 = 3 \left( C e^{3x} + \frac{2}{3} \right) - 2 \]

\[ = 3C e^{3x} + 2 - 2 \]

\[ = 3C e^{3x} \]

if \[ y(a) = 2 \]

\[ 2 = C e^0 + \frac{2}{3} \]

\[ y = \frac{4}{3} e^{3x} + \frac{2}{3} \]

\[ 2 = C + \frac{2}{3} \]

\[ C = \frac{4}{3} \]