Autonomous Equations and Population Dynamics

The first order ode \( \frac{dy}{dt} = f(y) \) is called autonomous.

The RHS depends on \( y \) alone - not on the independent variable \( t \).

We have seen several of these - most significantly \( y' = ry \), the exponential growth/decay equation.

One interesting fact about autonomous equations is that their direction fields are easy to draw because \( f \) does not depend on \( t \).

Exponential Growth: \( y' = ry \)
- simple model that works well for populations with no impediments to growth.
- of course, growth cannot occur indefinitely.

Logistic Growth.

Let's modify the exponential growth eq. so that

1. \( y' \approx ry \) when \( y \) is small
2. \( y' \rightarrow 0 \) as \( y \rightarrow K \) where \( K \) is the population that can be sustained.

\[ y' = r\cdot y \left( 1 - \frac{y}{K} \right) \]
- close to 1 when \( y \) is close to zero
- close to 0 when \( y \) is close to \( K \)
- positive if \( y < K \), negative if \( y > K \)
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This gives the logistic growth equation

$$\frac{dy}{dt} = r(1-\frac{y}{K})y$$

- $r$ is the intrinsic growth rate
- $K$ is the saturation level or environmental carrying capacity

Ex. \[ \frac{dy}{dt} = \frac{1}{2} (1-\frac{y}{1}) y = \frac{1}{2}(1-y)y \quad r = \frac{1}{2} \quad K = 1 \]

The direction field for this equation looks like

Consider \[ \frac{dy}{dt} = r(1-\frac{y}{K})y, \quad r > 0, \quad K > 0 \]

If $y > 0$ and $y < K$ then $r(1-\frac{y}{K}) > 0 \Rightarrow y$ increases
If $y > 0$ and $y > K$ then $r(1-\frac{y}{K})y < 0 \Rightarrow y$ decreases

So, the solution $y(t)$ will approach $K$ as $t \to \infty$
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Given \( \frac{dy}{dt} = f(y) = r\left(1 - \frac{y}{K}\right)y \) we can plot \( f(y) \) vs. \( y \)

Equilibrium point \( \frac{dy}{dt} = 0 \) (same as critical point)

Value of \( y \) for which \( \frac{dy}{dt} \) is positive, \( y \) is decreasing
Value of \( y \) for which \( \frac{dy}{dt} \) is negative meaning \( y(t) \) is increasing

This is why \( K \) is called the saturation level: If \( y \) is above it, \( y \) will decrease towards it. If \( y \) is below it, \( y \) will increase towards it.

The points 0 and \( K \) are called critical points.

The critical points in these problems can be characterized as being asymptotically stable or unstable
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To solve the logistic equation \( \frac{dy}{dt} = r(1 - \frac{y}{K})y \), we separate variables:

\[
\frac{dy}{(1 - \frac{y}{K})y} = r \, dt
\]

To integrate the LHS, we can use partial fractions:

\[
\frac{1}{(1 - \frac{y}{K})y} = \frac{A}{1 - \frac{y}{K}} + \frac{B}{y} \Rightarrow A = \frac{1}{K}, \quad B = 1
\]

So,

\[
\int \frac{dy}{K(1 - \frac{y}{K})} + \int \frac{dy}{y} = r \int dt
\]

\[-\ln|K-y| + \ln|y| = rt + C
\]

\[-\ln|\frac{y}{K}| = rt + C
\]

\[\frac{y}{K} = C e^{rt}
\]

If \( y(0) = y_0 \) then \( \frac{y_0}{K-y_0} = C \), so \( \frac{y}{K} = \frac{y_0}{K-y_0} e^{rt} \).

Solving for \( y \):

\[
\frac{1}{y} = \frac{y_0}{K-y_0} e^{-rt} \Rightarrow \frac{K-y}{y} = \frac{y_0}{K-y_0} e^{-rt}
\]

\[
\frac{K}{y} = \frac{(K-y_0) e^{-rt} + y_0}{y_0} \Rightarrow \frac{y}{K} = \frac{y_0}{y_0 + (K-y_0) e^{-rt}}
\]

\[
y = \frac{K y_0}{y_0 + (K-y_0) e^{-rt}}
\]

Notice that as \( t \to \infty \), \( y \to K y_0/y_0 = K \) if \( r > 0 \).
Ex. Let \( \frac{dy}{dt} = f(y) \) be \( \frac{dy}{dt} = 2y(1- \frac{e^y}{2}) \) \(-\infty < y < \infty\)
and sketch \( f(y) \) vs. \( y \). Find the critical points
and classify them as asymptotically stable or
unstable.

\[
\begin{align*}
0 & = 2y(1- \frac{1}{2}e^y) \\
& \Rightarrow y = 0 \text{ and } y = \ln 2 \approx 0.6931
\end{align*}
\]

\( y = 0 \) is an unstable critical point since
\( f(y) > 0 \) when \( y < 0 \), \( f(y) < 0 \) when \( y > 0 \)

\( y = \ln 2 \approx 0.6931 \) is an asymptotically stable critical point since \( f(y) > 0 \) when \( y < \ln 2 \)
and \( f(y) < 0 \) when \( y > \ln 2 \).