8.1 Euler's Method (Tangent Line Method)

Consider the IVP
\[
\begin{align*}
\frac{dy}{dt} &= f(t, y) \\
y(t_0) &= y_0
\end{align*}
\]

We will consider this solved if, on the interval in which the solution exists, we can obtain the value of \( y \) for any particular value of \( t \).

Example (Textbook) \( y' = \frac{1-t+4y}{f(t, y)} \), \( y(0) = 1 \)

Let's try and construct the solution graphically.

Basic idea is that we step from our initial point a distance \( h \) in the horizontal direction and \( hf(t_0, y_0) \) in the vertical direction.

\( y(0.25) = y(0) + hf(0, y(0)) \)

Euler's Method

Given \( t_0 \) and \( y_0 \) and stepsize \( h \)

\[
\begin{align*}
y_{k+1} &= y_k + h f(t_k, y_k) \\
t_{k+1} &= t_k + h = t_0 + (k+1)h
\end{align*}
\]
Notice that we can obtain the formula for Euler's Method using a Taylor Series:

\[ y(t_{n+1}) = y(t_n) + y'(t_n) \cdot h + y''(\xi_n) \frac{h^2}{2!} \]

- Remainder term
- for some \( \xi_n \) between \( t_n \) and \( t_{n+1} \)

If we ignore the last term \( y''(\xi_n) \frac{h^2}{2!} \) then we see that we have the Euler iteration formula:

\[ y(t_{n+1}) = y(t_n) + h f(t_n, y_n) \]

or

\[ y_{n+1} = y_n + h f(t_n, y_n) \]

There are other ways to generate this formula as well.
Consider \( y' = 3 - 2y \) autonomous, 1\(^{st}\) order, linear.

Solution \( y = -\frac{3}{2}(e^{-2t} - 1) + y_0 e^{-2t} \)

Applying Euler method with \( h = 0.1 \), \( y_0 = 1 \)

<table>
<thead>
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