3.2 Fundamental Solutions of Linear Homogeneous Equations

We now seek to answer some basic questions regarding 2nd order linear o.d.e.s.

1) When does the o.d.e. have a solution?
2) If we find a solution - are there others?

Begin by introducing the Second Order Differential Operator

\[ L[\phi] = \phi'' + p(x)\phi' + q(x)\phi \]

Where \( p \) and \( q \) are continuous on open interval \( I : \alpha < x < \beta \)
and \( \phi \) is twice-differentiable on \( I \).

So, the general Second order linear Initial value problem is

Find \( y \) s.t. \[ L[y] = y'' + p(x)y' + q(x)y = g(x) \]
Subject to \( y(\alpha) = y_0, \quad y'(\alpha) = y'_0 \)

\[ \text{(1)} \]

Thm 3.2.1

If \( p, q \) and \( g \) are continuous on an open interval \( I \), then there exists exactly one solution \( y = \phi(x) \) of \( \text{(1)} \), and the solution exists throughout \( I \).

Note that the o.d.e. must be in the form of \( \text{(1)} \), with the coefficient of \( y'' \) equal to 1.
Thm 3.2.2 (principle of superposition) (linear combination)

If \( y_1 \) and \( y_2 \) are two solutions of the differential equation

\[
L[y] = y'' + p(x)y' + q(x)y = 0
\]

then the linear combination \( c_1 y_1 + c_2 y_2 \) is also a solution for any values of the constants \( c_1 \) and \( c_2 \).

**Proof**

\[
L[c_1 y_1 + c_2 y_2] = \frac{d^2}{dx^2} (c_1 y_1 + c_2 y_2) + p(x) \frac{d}{dx} (c_1 y_1 + c_2 y_2) + q(x) (c_1 y_1 + c_2 y_2) = 0
\]

\[
= c_1 y_1'' + c_2 y_2'' + p(x) c_1 y_1' + p(x) c_2 y_2' + q(x) c_1 y_1 + q(x) c_2 y_2 = 0
\]

\[
= c_1 \left( y_1'' + p(x)y_1' + q(x)y_1 \right) + c_2 \left( y_2'' + p(x)y_2' + q(x)y_2 \right) = 0
\]

\[
= c_1 L[y_1] + c_2 L[y_2] = 0
\]

\[
= c_1 (0) + c_2 (0) = 0
\]

This works because \( L \) is a **linear operator**.

So...

Assuming certain things about the d.e., we know

1) that a unique solution will exist, given specific initial conditions.

2) that a linear combination of functions which satisfy the d.e. will also satisfy the d.e.

\[\text{homogeneous}\]
Given suitable I.C. and properties of the diff. eq.
we now consider the question

"Can I find \( C_1 \) and \( C_2 \) such that \( y = C_1 y_1(x_0) + C_2 y_2(x_0) \)
satisfies the initial conditions?"

1. \( y(x_0) = y_0 \), so \( y_0 = C_1 y_1(x_0) + C_2 y_2(x_0) \)
2. \( y'(x_0) = y'_0 \), so \( y'_0 = C_1 y'_1(x_0) + C_2 y'_2(x_0) \)

2 eq. with 2 unknowns

\[ y_1(x_0) C_1 + y_2(x_0) C_2 = y_0 \]
\[ y'_1(x_0) C_1 + y'_2(x_0) C_2 = y'_0 \]

Using Cramer's rule we find

\[
C_1 = \frac{\begin{vmatrix} y_0 & y_2(x_0) \\ y'_0 & y'_2(x_0) \end{vmatrix}}{\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y'_1(x_0) & y'_2(x_0) \end{vmatrix}} \\
C_2 = \frac{\begin{vmatrix} y_1(x_0) & y_0 \\ y'_1(x_0) & y'_0 \end{vmatrix}}{\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y'_1(x_0) & y'_2(x_0) \end{vmatrix}}
\]

Which say that \( C_1 \) and \( C_2 \) do indeed exist
provided the determinant in the denominator is not zero.

\[
W = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y'_1(x_0) & y'_2(x_0) \end{vmatrix} = y_1(x_0) y'_2(x_0) - y'_1(x_0) y_2(x_0)
\]

This determinant is called the "Wronskian"
Thm 3.2.3 (page 124) (pg 141 7th)

Main Thrust → if $W \neq 0$ at $x_0$, where the initial conditions are supplied, then $c_1$ and $c_2$ can be found for which $y = c_1y_1 + c_2y_2$ satisfies (X) (the d.e. and the i.c.s).

Ex: Recall $y'' - 4y = 0 \quad \Rightarrow \quad y = c_1e^{2x} + c_2e^{-2x}$

$y(0) = 1 \quad \Rightarrow \quad c_1 = \frac{1}{2} \quad , \quad c_2 = \frac{1}{2}$

$y(0) = 0$

have $y_1 = e^{2x} \quad , \quad y_2 = e^{-2x}$

\[
W = \begin{vmatrix} y_1(x) & y_2(x) \\ y'_1(x) & y'_2(x) \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix} = -2 - 2 = -4
\]

$W = \begin{vmatrix} y_1(x) & 0 \\ y'_1(x) & -4 \end{vmatrix} = \frac{1(-2) - 0}{-4} = \frac{1}{2}$ \quad √

$C_1 = \begin{vmatrix} 1 \\ y_1(x) \\ y'_1(x) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 2 & -4 \end{vmatrix} = \frac{1}{2}$ \quad √

→ The solution $y = c_1y_1 + c_2y_2$, is called the general solution since any solution of the d.e. is contained within this family of functions, i.e. a linearly independent set of solutions span the set of all solutions.

→ If $W \neq 0$, $y_1$ and $y_2$ form a fundamental set of solutions.
THM 3.2.4

If \( y_1 \) and \( y_2 \) are two solutions of the differential equation
\[ L[y] = y'' + p(x)y' + q(x)y = 0, \]
and if there is a point \( x_0 \) where the Wronskian of \( y_1 \) and \( y_2 \) is nonzero, then the family of solutions \( y = c_1 y_1 + c_2 y_2 \)
with arbitrary \( c_1 \) and \( c_2 \) includes every solution of the ODE.

Proof:

Assume \( \phi(x) \) is any function which satisfies the ODE
and initial conditions (\( k \)). Thus \( \phi(x_0) = y_0 \) and
\[ \phi'(x_0) = y_0'. \]

Now show that \( \phi(x) \) is contained in \( y = c_1 y_1 + c_2 y_2 \).

Since by Thm 3.2.2, \( y = c_1 y_1 + c_2 y_2 \) is a solution of the ODE.
and since
\[ W = y_1(x)y_2'(x) - y_1'(x)y_2(x) \neq 0. \]
we know, by Thm 3.2.3 that there exist \( c_1 \) and \( c_2 \) which allow \( y \)
to solve (\( k \)).

By Thm 3.2.1, if \( \phi(x) \) and \( y = c_1 y_1 + c_2 y_2 \) both solve
(\( k \)) then \( \phi(x) = c_1 y_1 + c_2 y_2 \) since the solution is unique.

Thus \( \phi \) is contained in \( y = c_1 y_1 + c_2 y_2 \).
Please note that while Theorem 3.2.4 says $y$ and $y_2$ are fundamental solutions, from a basis for all solutions, it is possible that solutions appear not to depend on these functions.

Ex. Recall $y'' - 4y = 0$, $y(0) = 1$, $y'(0) = 0$; had the solution

$$y = \frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x}.$$  

Well, $y = C_1\cosh 2x + C_2\sinh 2x$ also satisfies the o.d.e.

$$\cosh u = \frac{e^u + e^{-u}}{2}, \quad \sinh u = \frac{e^u - e^{-u}}{2}.$$  

Finding $C_1$ & $C_2$, we match i.e., and get

$$y = \cosh 2x$$ is the solution.