3.5 Repeated Roots; Reduction of Order

Consider \( y'' - 2y' + y = 0 \) \( \Rightarrow r^2 - 2r + 1 = 0 \)

\[(r - 1)(r - 1) = 0 \quad \text{so } r = 1 \text{ is a repeated root.} \]

Clearly \( y_1 = e^x \) is a solution. But \( y_2 = e^x \) is the same – i.e. \( y_1 \) and \( y_2 \) do not form a linearly independent set of solutions – the Wronskian of \( y_1 \) and \( y_2 \) is zero.

Let us try another approach: if \( e^x \) is a solution, we already know that \( ce^x \) is also a solution. Perhaps \( \psi(x)e^x \) will also be a solution – we can put it into the o.d.e. and find what \( \psi(x) \) needs to be.

\[
y = \psi(x)e^x \quad y' = \psi'e^x + \psi e^x = (\psi' + \psi)e^x \\
y'' = \psi''e^x + \psi'e^x + \psi e^x + \psi e^x = (\psi'' + 2\psi' + \psi)e^x
\]

So

\[
(\psi'' + 2\psi' + \psi)e^x - 2(\psi' + \psi)e^x + \psi e^x = 0 \\
\psi''e^x + (2\psi' - 2\psi)e^x + (\psi - 2\psi + \psi)e^x = 0 \\
\psi''e^x = 0
\]

\[
\psi'' = 0 \\
\psi' = C_1 \\
\psi = C_1xe^x + C_2
\]

So we find that \( y = (C_1xe^x + C_2)e^x = C_1xe^x + C_2e^x \) is a solution.

Actually, this is our general solution, the fundamental solution set being \( xe^x \) and \( e^x \).
Given \( ay'' + by' + cy = 0 \)

Let \( y = e^{rt} \) so \( ar^2 + br + c = 0 \)

If \( r = \alpha \) is a repeated root then \( y_1 = e^{\alpha t} \) is a solution. Let \( y = V(t)e^{\alpha t} \)

\( y' = Ve^{\alpha t} + \alpha Ve^{\alpha t} \)
\( y'' = V'e^{\alpha t} + 2\alpha Ve^{\alpha t} + \alpha^2 Ve^{\alpha t} \)

\( ay'' + by' + cy = a(V'' + 2\alpha V' + \alpha^2 V)e^{\alpha t} + b(V' + \alpha V)e^{\alpha t} + cVe^{\alpha t} = 0 \)

\( = V''ae^{\alpha t} + V'(2\alpha a + b)e^{\alpha t} + V(a^2 + \alpha b + c)e^{\alpha t} = 0 \)

\( \alpha^2 + \alpha b + c = 0 \) since \( \alpha = \text{a root} \).

If a quadratic eq has repeated roots it must have a horizontal tangent at \( y = 0 \) so \( \frac{d}{dr}(ar^2 + br + c) = 0 \) at \( \alpha \): \( \frac{d}{dr}(ar^2 + br + c) \bigg|_{\alpha} = 2\alpha a + b \bigg|_{\alpha} = 2\alpha a + b = 0 \)

\( \therefore V''ae^{\alpha t} = 0 \). We require \( a \neq 0 \) so \( V'' = 0 \)

\( \Rightarrow V = c_1 t + c_2 \)

\( y = c_1 t + c_2 e^{\alpha t} = c_1 te^{\alpha t} + c_2 e^{\alpha t} \) is also a sol'n.

\( \text{already known} \)
\( \text{new, linearly independent sol'n} \).
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To make sure, \[ W = \begin{vmatrix} xe^x & e^x \\ (x+1)e^x & e^x \end{vmatrix} = xe^{2x} - (x+1)e^{2x} = e^{2x} \neq 0 \]

General Rule: If we get repeated roots of the characteristic eq, say \( r = \lambda \), then a fundamental set of solution is \( e^{\lambda x} \) and \( xe^{\lambda x} \).

So, given \( ay'' + by' + cy = 0 \):

1. Solve characteristic eq \( ar^2 + br + c = 0 \) for roots \( r_1 \) and \( r_2 \).

2. Case 1: \( r_1 \neq r_2 \), real
   \[ \Rightarrow y = c_1 e^{r_1 x} + c_2 e^{r_2 x} \]

   Case 2: \( r_1 = \lambda + i\mu, \ r_2 = \lambda - i\mu \)
   \[ \Rightarrow y = c_1 e^{\lambda x} \cos \mu x + c_2 e^{\lambda x} \sin \mu x \]

   Case 3: \( r_1 = r_2 = r \)
   \[ \Rightarrow y = c_1 xe^{rx} + c_2 e^{rx} \]

Notice that both cases 1 and 3 involve exponentials which can go to zero or to infinity. Note that relatively few, if any, sign changes occur in the solution as \( x \to \infty \).

This is in contrast to case 2 which has periodic functions as part of the solution.
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Reduction of Order

If we know one solution of $y'' + py' + qy = 0$
we can find another independent solution using a similar
method - if $y_1$ is a solution then seek another solution
of the form $vy_1$.

$y_1$ satisfies $y'' + py' + qy = 0$

$y_2 = vy_1$
$y_1' = vy_1' + vy_1$
$y_2' = vy_1' + vvy_1' + vy_1''$

Ode becomes

$v''y_1 + 2vy_1' + vy_1'' + p(vy_1 + vy_1') + q(vy_1) = 0$
$v''y_1 + v'(2y_1' + py_1) + v(y_1'' + py_1' + qy_1) = 0$

$v''y_1 + v'(2y_1' + py_1) = 0 \quad (\ast)$

If $w = v'$

$p(x) = 2\frac{y_1'}{y_1} + p(x) \Rightarrow w' + p(x)w = 0$

$\frac{dw}{w} = -p(x)dx$

$\ln w = -\int p(x)dx$

$w = v' = e^{-\int p(x)dx}$

integrate to find $v(x)$

Ex.

$x^2y'' + 2xy' - 2y = 0 \quad x > 0$

$y_1(x) = x$

$y_2(x) = y(x) x$

$p(x) = \frac{2}{x}$
$q(x) = -\frac{2}{x^2}$

$y'_1 = v'x + v$
$y''_1 = v''x + v' + v' = v''x + 2v'$

From $(\ast)$

$v''x + v'(2 + \frac{2}{x}) = 0 \Rightarrow \dot{x}v'' + 4v' = 0$
\[ W' - \frac{4}{x} w = -4 \frac{dx}{w} \]

\[ \ln |w| = -4 \ln |x| + C \]

\[ W = C x^{-4} \]

\[ V = \int C x^{-4} dx = C_1 x^{-3} + C_2 \]

So \[ y_2 = x^{-1} x^{-3} = x^{-2} \] is also a solution.

Check:

\[ x^{-2} (6x^{-4}) + 2x (-2x^{-3}) - 2x^{-2} = 6x^{-2} - 4x^{-2} - 2x^{-2} = 0 \]

General solution would be \[ y = C_1 x + C_2 x^{-2} \]

For discussion #33, 34

**Critically Damped Harmonic Oscillator**

\[ m \ddot{x} + \gamma \dot{x} + kx = 0 \]

\[ \gamma \text{ is coefficient of friction, } \gamma > 0 \]

Assume \[ x = e^{rt} \Rightarrow mr^2 + \gamma r + k = 0 \Rightarrow r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m} \]

If \( \gamma^2 > 4km \) -> two real distinct roots:

\[ x = C_1 e^{\frac{\gamma - \sqrt{\gamma^2 - 4km}}{2m} t} + C_2 e^{\frac{\gamma + \sqrt{\gamma^2 - 4km}}{2m} t} \]

Exponents are both negative so solutions decay exponentially.

If \( \gamma^2 < 4km \) -> complex conjugate roots:

\[ r = \frac{-\gamma}{2m} \pm \frac{i \sqrt{4km - \gamma^2}}{2m} \]

\[ x = C_1 e^{\frac{\gamma}{2m} t} \cos \omega t + C_2 e^{\frac{\gamma}{2m} t} \sin \omega t \]

Decay since \( \omega > 0 \)

If \( \gamma^2 = 4km = 0 \) -> real repeated roots \[ r = -\frac{\gamma}{2m} \]

\[ x = C_1 e^{\frac{-\gamma}{2m} t} + C_2 e^{\frac{-\gamma}{2m} t} \]

Decay, no oscillation -

Slowest decay w/o osc. \[ \Rightarrow \text{Critically damped} \]