4.2 Homogeneous Equations with Constant Coefficients

\[ L[y] = a_0 y^{(m)} + a_1 y^{(m-1)} + \ldots + a_n y = 0 \quad a_i \text{ real} \]

Assume a solution of the form \( y = e^{rx} \)

\[ L[e^{rx}] = a_0 r^m e^{rx} + a_1 r^{m-1} e^{rx} + \ldots + a_n e^{rx} = 0 \]
\[ = e^{rx} \left( a_0 r^m + a_1 r^{m-1} + \ldots + a_n r + a_0 \right) = 0 \]

So
\[ a_0 r^m + a_1 r^{m-1} + \ldots + a_n r + a_0 = 0 \]

is the characteristic eq.

We need to find the roots of this polynomial. The roots will be

- real and distinct
- real and repeated
- complex conjugates (distinct or repeated)

or combinations of these. We expect to find \( n \) roots, counting multiplicities.

Once the roots \( r_1, \ldots, r_n \) are determined, we then can proceed with the construction of our solution.

If we arrange the roots so that the first \( m \), \( r_1, \ldots, r_m \) are real and distinct, then
\[ y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \ldots + c_m e^{r_m x} \]
is a solution, and if \( m = n \), this is the general solution of \( L y = 0 \).
It is important, as in the \( n=2 \) case, that \( e^{rx} \ i=1, \ldots, m \) are linearly independent. Since \( r_1, \ldots, r_m \) are distinct, we have the independence we need. See 4.2 #40 for proof.

Suppose that \( r_{m+1}, \ldots, r_{m+s} \) are a single root with multiplicity \( s \). Then if all these roots are called \( x \)

\[
y = C_{m1} e^{rx} + C_{m2} x e^{rx} + \cdots + C_{ms} x^{s-1} e^{rx}
\]

is also a solution of \( L[y] = 0 \). To prove this we need only show that \( x^p e^{rx} \), \( p \leq s \), is a solution since \( L[y] \) is a linear operator.

\[
L[e^{rx}] = e^{rx} (r-x)^s H(r) - \text{all other factors of characteristic polynomial.}
\]

Now diff. w.r.t. \( r \) on both sides, interchanging diff. w.r.t. \( r \) and \( x \).

\[
L[x e^{rx}] = x e^{rx} (r-x)^s H(r) + e^{rx} [s (r-x)^s + X^{s-1} H(r) + e^{rx} (r-x)^s H'(r) \]
\]

if \( s=x \)

\[
L[x e^{rx}] = e^{rx} \left[ x (x-x)^s H(r) + s (x-x)^s H'(r) + (x-x)^s H''(r) \right]
\]

\[
= 0 \quad \text{if } s \geq 2
\]

Diff. again to get \( L[x^2 e^{rx}] \) and sub. \( r=x \) to find

\[
L[x^2 e^{rx}] = 0 \quad \text{if } s \geq 3
\]

Continuing in this manner we see that \( x^p e^{rx} \) is a solution for any \( p \leq s \).
Now that all real roots have been dealt with, we consider the complex roots. Since the coeff of the ode are real we know that complex roots will occur in conjugate pairs.

For each distinct complex conjugate pair \( r = \lambda \pm \iota \mu \) both

\[
\begin{align*}
& e^{\lambda x} \cos \mu x \quad e^{\lambda x} \sin \mu x \\
& xe^{\lambda x} \cos \mu x \quad xe^{\lambda x} \sin \mu x \\
& x^2 e^{\lambda x} \cos \mu x \quad x^2 e^{\lambda x} \sin \mu x
\end{align*}
\]

are solution of \( L[y] = 0 \).

If any complex roots are repeated \( s \) times, then

\[
\begin{align*}
& e^{\lambda x} \cos \mu x \quad e^{\lambda x} \sin \mu x \\
& xe^{\lambda x} \cos \mu x \quad xe^{\lambda x} \sin \mu x \\
& x^2 e^{\lambda x} \cos \mu x \quad x^2 e^{\lambda x} \sin \mu x \\
& \ldots
\end{align*}
\]

are also solutions of \( L[y] = 0 \).

**Ex** \( y'' + 2y' - 8y + 5y = 0 \)

\[
\rightarrow r^2 + 2r + 2r^2 - 8r + 5 = 0
\]

\[
\rightarrow r = 1, 1, -1 + 2i, -1 - 2i \quad \text{derived on next page}
\]

\[
e^x, \quad xe^x, \quad e^x \cos 2x, \quad e^x \sin 2x
\]

general solution is

\[
y = c_1 e^x + c_2 xe^x + c_3 e^{-x} \cos \mu x + c_4 e^{-x} \sin \mu x
\]
\[ y'' + 2y' - 8y + 5y = 0 \quad \text{assume } y = e^{rx} \]

\[ r^4 + 2r^2 - 8r + 5 = 0 \]

leading coefficient is 1, factors are \pm 1
constant is 5, factors are \pm 1, \pm 5

possible rational roots are \( \pm \frac{1}{1}, \pm \frac{5}{1} \) or \( \pm 1, \pm 5 \)

\[ R = 1 \] is not a root: \( 1^4 + 2 \cdot 1^2 - 8 \cdot 1 + 5 = 0 \)
\[ R = -1 \] is not a root: \( 1 + 2 + 8 + 5 \neq 0 \)
\[ R = 5 \] is not a root
\[ R = -5 \] is not a root

divide \((r-1)\) out of polynomial

\[ \frac{r^4 + 2r^2 - 8r + 5}{r - 1} \]

so \( r^4 + 2r^2 - 8r + 5 = 0 \)

possible roots again are \( \pm 1, \pm 5 \)
\[ r = 1 \] is a root: \( 1 + 2 + 3 - 5 = 0 \)
\[ r = -1 \] is not a root: \(-1 + 2 - 3 - 5 \neq 0 \)
\[ r = \pm 5 \] is not a root

\[ \frac{r^2 + 2r + 5}{r - 1} \]

so \( r^2 + 2r + 5 = 0 \)

\[ r = \frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2i \]

\[ r = 1, -1 - 2i, -1 + 2i \] are our roots
Suggestions for finding roots of higher order polynomials:

1. Given \( a_0r^n + a_1r^{n-1} + \cdots + a_{n-1}r + a_n = 0 \)

   If \( r = \frac{p}{q} \) is a rational root in simplest form, then \( p \) is a factor of \( a_n \) (constant term coefficient) and \( q \) is a factor of \( a_0 \) (leading coefficient).

2. Once one root \( x \) is found, the polynomial can be reduced by dividing by \( (x-x) \)

\[
\frac{P(x)}{(x-x)} \rightarrow \text{new } P_{n-1}(x)
\]

3. Graphing or root-finding tools also will work.