4.3 Method of Undetermined Coefficients

If \( L[y] = g(x) \), and if \( L \) has constant coefficients, we can use the method of undetermined coefficients if \( g(x) \) is of an appropriate form.

\( g(x) \) must be a sum and/or product of

1. polynomials of \( x \)
2. exponentials in \( x \)
3. sines and cosines in \( x \)

1) Polynomial part. If \( g(x) \) is a polynomial of degree \( m \) then we can assume a particular solution of the form \( y(x) = A_0 x^m + A_1 x^{m-1} + \ldots + A_m \).

Problem:

\[ y'' - y'' = x \quad \Rightarrow \quad r^2 - r = 0 \quad \Rightarrow \quad r = 0, 0, 1, 1 \]

\[ y_c = C_1 + C_2 x + C_3 e^x + C_4 e^{-x} \]

Now, since \( g(x) = x \), we assume a particular solution of the form \( y(x) = A_1 x + A_0 \). However, this is part of the general solution - \( y'' \) and \( y'' \) are both zero. Maybe \( y(x) = A_1 x^2 + A_0 x \)? No - still doesn't give \( x \).

Try \( y(x) = A_1 x^2 + A_0 x^1 \)

\[ 0 - 6A_1 x - A_0 = x \quad \Rightarrow \quad A_1 = -\frac{1}{6}, \quad A_0 = 0 \]

So, if \( r = 0 \) is an \( s \)-fold root of the characteristic eq. \( [s \text{ is the order of the lowest derivative of } y \text{ in the eq}] \), then \( y(x) = x^5 [A_0 x^m + A_1 x^{m-1} + \ldots + A_m] \) is a part. soln.
Now consider exponents of $x$. If $g(x) = P(x)e^{\lambda x}$ where $P(x)$ is a polynomial of degree $m$. We assume a solution of the form

$$Y(x) = e^{\lambda x} \left[ A_0 x^m + A_1 x^{m-1} + \ldots + A_m \right]$$

Which is fine unless $\lambda$ is a root of the characteristic eq.

$$y'' - y = xe^x \quad \Rightarrow \quad \lambda = 0, 1, 2 \quad \Rightarrow \quad y_c = c_1 e^0 + c_2 xe^x + c_3 e^x$$

So we can't assume $Y = [A_0 x + A_1]e^x$, since it's a solution of the homogeneous eq.

Try $Y(x) = x [A_0 x + A_1]e^x = A_0 x^2 e^x + A_1 xe^x$

$$Y' = A_0 x^2 e^x + 2A_0 x e^x + A_1 xe^x + A_1 e^x$$

$$Y'' = A_0 x^2 e^x + 4A_0 x e^x + 2A_0 e^x + A_1 xe^x + 2A_1 e^x$$

$$Y''' = A_0 x^2 e^x + 6A_0 x e^x + 6A_0 e^x + A_1 xe^x + 3A_1 e^x$$

$$Y'''' = A_0 x^2 e^x + 8A_0 x e^x + 12A_0 e^x + A_1 xe^x + 4A_1 e^x$$

$$y'''' - y'' = 4A_0 x e^x + 10A_0 e^x + 2A_1 e^x = xe^x$$

$$4A_0 = 1 \quad 10A_0 + 2A_1 = 0$$

$$A_0 = \frac{1}{4} \quad A_1 = -\frac{1}{2}$$

$A_0 = \frac{1}{4}$, $A_1 = -\frac{1}{2}$, $\lambda_1 = -\frac{1}{2}$

$Y = \frac{1}{4} [x^2 - 5x]e^x$ is a particular soln.
4.5

Now if \( g(x) = P_m(x) e^{ax} [K_1 \sin bx + K_2 \cos bx] \)

we assume \( Y(x) = \left[ A_0 x^n + A_1 x^{n-1} + \ldots + A_m \right] e^{ax} \sin bx \)
\( + \left[ B_0 x^n + B_1 x^{n-1} + \ldots + B_m \right] e^{ax} \cos bx \)

but will need additional factor of \( x \) if \( a \pm ib \) are roots of the homogeneous eq.

See table on top of page \( 204 \) th \( 219 \) th \( 175 \) th \( 161 \) th

Method of Annihilators

We will use \( D = \frac{d}{dx} \). Then
\[ y'' - y = (D^2 - 1)y \]
\[ y''' + 3y'' + 3y = (D^3 + 3D^2 + 3)y \]

Ex. \( y''' - 2y'' + y' = x^3 + 2e^x \)

\[ (D^3 - 2D^2 + 1)y = D(D^2 - 2D + 1)y = D(D - 1)^2 y = x^3 + 2e^x \]

Suppose we knew what operators to apply so that the RHS became zero, giving us a homogeneous eq.

How do we get \( x^3 \to 0 \)? use \( D^4 \)
\( 2e^x \to 0 \)? use \( D - 1 \)

So, \( D^4(D - 1) \) will "annihilate" \( x^3 + 2e^x \)
\[ D^4(D-1)(x^3+2e^x) = (D^5-D^4)(x^3+2e^x) \]
\[ = D^5x^3-D^4x^3+2D^5e^x-2D^4e^x \]
\[ = 0-0+2e^x-2e^x = 0 \]

\[ \therefore D^4(D-1) \cdot D(D-1)^2 y = 0 \]
\[ D^5(D-1)^3 y = 0 \]

characteristic polynomial:
\[ r^5(r-1)^3 = 0 \rightarrow r = 0, 0, 0, 0, 0, 1, 1, 1 \]

\[ S_0, \quad y = c_1 + c_2 x + c_3 x^3 + c_4 x^3 + c_5 x^4 + c_6 x^5 + c_7 x^6 + c_8 x^7 e^x \]

Complementary Solution:

particular solution

Note that this method exchanges a nonhomogeneous differential equation for a higher order homogeneous equation. This new equation is not necessarily any easier to solve...
Ex. \( y'' + 4y'' = \sin 2x + xe^x + 4 \)

\[ D^2(D^2+4)y = \sin 2x + xe^x + 4 \]

Find annihilator for \( \sin 2x, xe^x, 4 \)

\[
\begin{align*}
\sin 2x : & \quad D^2+4 \\
x e^x : & \quad (D-1)^2 \\
y : & \quad D
\end{align*}
\]

So

\[
D(D^2+4)(D-1)^2 D^2(D^2+4)y = D(D^2+4)(D-1)^2 (\sin 2x + xe^x + 4)
\]

\[
D^3(D^2+4)^2(D-1)^2 y = 0
\]

\[
(r^2+4)^2(r-1)^2 = 0 \quad \Rightarrow \quad r = 0, 0, 0, \pm 2i, \pm 2i, 1, 1
\]

\[
y(x) = C_1 + C_2 x + C_3 x^2 + C_4 \sin 2x + C_5 \cos 2x + C_6 x \sin 2x + C_7 x \cos 2x + C_8 e^x + C_9 xe^x
\]

Char. eq. of orig. eq. is \( r^2(r^2+4) = 0 \) \( \Rightarrow \) \( r = 0, 0, \pm 2i \)

So \( 1, x, \sin 2x, \cos 2x \) are solns of homogeneous problem \( \ldots \)

\[
y(x) = A_1 x^2 + x(B_1 \sin 2x + B_2 \cos 2x) + (C_1 + C_2 x) e^x
\]

Is a particular soln.

Find \( A_1, B_1, B_2, C_1, C_2 \) by substituting this back into the original equation.