5.1 Review of Power Series

A power series has the form \( \sum_{n=0}^{\infty} a_n (x-x_0)^n \) where \( x_0 \) is a fixed point.

The power series converges at \( x \) if
\[
\lim_{m \to \infty} \sum_{n=0}^{m} a_n (x-x_0)^n
\]
exists.

The power series converges absolutely at \( x \) if the series
\[
\sum_{n=0}^{\infty} |a_n (x-x_0)^n|
\]
converges.

If a power series converges absolutely at \( x \) then it converges at \( x \). The converse is not true in general.

The two most frequently used tests for convergence are the \( n^{th} \) root test and the ratio test.

**\( n^{th} \) root test:** The series \( \sum_{n=0}^{\infty} a_n (x-x_0)^n \) converges absolutely at \( x \) if
\[
\lim_{n \to \infty} \sqrt[n]{|a_n (x-x_0)^n|} = L < 1
\]
This reduces to
\[
|x-x_0| \lim_{n \to \infty} \sqrt[n]{|a_n|} = L < 1
\]
5.1

**Ratio Test:** The series \( \sum_{n=0}^{\infty} a_n (x-x_0)^n \) converges absolutely at \( x \) if

\[
\lim_{n \to \infty} \left| \frac{a_{n+1} (x-x_0)^{n+1}}{a_n (x-x_0)^n} \right| = L < 1
\]

or

\[
|x-x_0| \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1
\]

(Note: this test only applies if all \( a_n \neq 0 \), \( n > N \) for some \( N \))

In each case \( L \) is a constant. If \( L < 1 \) then we have absolute convergence. If \( L > 1 \) we know that the series diverges. If \( L = 1 \) the test is inconclusive.

For any power series there is a nonnegative number \( \rho \) called the **radius of convergence** such that if

\[
|x-x_0| < \rho
\]

then the series converges absolutely and if

\[
|x-x_0| > \rho
\]

the series diverges. It may either converge or diverge at

\[
|x-x_0| = \rho
\]

**Example:** Determine the radius of convergence of

\[
a) \sum_{n=0}^{\infty} \frac{n}{2^n} x^n \quad b) \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}
\]
a) \[ \sum_{n=0}^{\infty} \frac{n}{2^n} x^n \] We will use the ratio test.

\[ \lim_{n \to \infty} \left| \frac{\frac{n+1}{2^{n+1}} x^{n+1}}{\frac{n}{2^n} x^n} \right| = \left| \frac{n+1}{2n} \right| \]

\[ = |x| \lim_{n \to \infty} \left| \frac{n+1}{2n} \right| = \frac{1}{2} |x| \lim_{n \to \infty} \left| \frac{n+1}{n} \right| \]

\[ = \frac{1}{2} |x| \]

We want this to be less than one, so

\[ \frac{1}{2} |x| < 1 \]

\[ |x| < 2 \]

\[ |x| = 2 \]

The interval of convergence is \( x \in [-2, 2] \) but may also include either endpoint.

Check \( x = -2 \): \[ \sum_{n=0}^{\infty} \frac{n}{2^n} (-2)^n = \sum_{n=0}^{\infty} (-1)^n \cdot n \]

Diverges since terms don't go to 0.

Check \( x = +2 \): \[ \sum_{n=0}^{\infty} \frac{n}{2^n} (2)^n = \sum_{n=0}^{\infty} n \]

Diverges since terms don't go to 0.

\[ \text{Interval of convergence: } -2 < x < 2 \]
b) \[ \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \]

\[ \lim_{n \to \infty} \left| \frac{x^{2(n+1)}}{(n+1)!} \frac{n!}{x^{2n}} \right| \]

\[ = \lim_{n \to \infty} \left| x^2 \frac{n!}{(n+1)n!} \right| \]

\[ = x^2 \lim_{n \to \infty} \left| \frac{1}{n+1} \right| = 0 \]

Since this limit is zero for all \( x \), the radius of convergence is infinite:\[ p = \infty \]

Now, assume

\[ f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n \]

\[ g(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^n \]

(note: \( n \) is a "dummy parameter", much like the variable of integration in a definite integral)

In this case we say the series \( \sum_{n=0}^{\infty} a_n (x-x_0)^n \) converges to \( f(x) \). There may be restrictions on the interval - well, assume \( |x-x_0| < \rho \)

\[ \rho > 0 \]

1. \( f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) (x-x_0)^n \)

(Convergent) Series may be added term-by-term.
\[ f(x) g(x) = \left[ \sum_{n=0}^{\infty} a_n (x-x_0)^n \right] \left[ \sum_{n=0}^{\infty} b_n (x-x_0)^n \right] \]

\[ = \sum_{n=0}^{\infty} c_n (x-x_0)^n \]

\[ c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 \]

Division is also defined whenever \( g(x) \neq 0 \).

3. If and all its derivatives exist on \( |x-x_0| < \rho \) and
\[ a_n = \frac{f^{(n)}(x_0)}{n!} \]

The Taylor series for a function \( f(x) \) is
\[ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n \]

[\( f \) is analytic at \( x_0 \)]

4. If \( \sum a_n (x-x_0)^n = \sum b_n (x-x_0)^n \) for each \( x \)
then \( a_n = b_n \) for \( n=0, 1, 2, \ldots \)
It is often useful to "shift" the index of summation (remember that it's just a dummy parameter).

\[ \sum_{n=2}^{\infty} a_n (x-x_0)^n = a_2 (x-x_0)^2 + a_3 (x-x_0)^3 + \ldots \]

But...

\[ \sum_{n=0}^{\infty} a_{n+2} (x-x_0)^{n+2} = a_2 (x-x_0)^2 + a_3 (x-x_0)^3 + \ldots \]

We can always accomplish this by

1. Replacing \( n \) (or whatever the index of summation happens to be) by \( n + a \) where \( a \) is the amount we want to shift by.

**VIN:** (Very Important Note) always use parentheses around the \( n \) that's replaced.

**Ex**

Change \( \sum_{n=1}^{\infty} a_n (x-x_0)^{2n} \) so that the index of summation begins at \( 0 \).

\[ \sum_{n+1=1}^{\infty} a_{n+1} (x-x_0)^{2(n+1)} \]

\[ \sum_{n=0}^{\infty} a_{n+1} (x-x_0)^{2n+2} \quad \text{Done.} \]