Starting with \( y'' + p(x) y' + q(x) y = 0 \) and assuming \( x = 0 \) is a regular singular point, we multiply by \( x^\ell \) to get

\[
x^\ell y'' + x [x p(x)] y' + [x^2 q(x)] y = 0 \quad (\ast)
\]

which resembles the Euler equs \( x^2 y'' + d x y' + \beta y = 0 \). Since \( x = 0 \) is a regular sing. pt, we know

\[
x^\ell p(x) = \sum_{n=0}^{\infty} p_n x^n \quad x^\ell q(x) = \sum_{n=0}^{\infty} q_n x^n \quad (\ast \ast)
\]

have series expansion with non-zero radius of convergence.

We assume \( y = x^\ell \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r} \); \( a_0 \neq 0 \)

and substitute into \( (\ast) \) with \( (\ast \ast) \):

\[
\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} \sum_{m=0}^{\infty} p_m x^m + \sum_{n=0}^{\infty} a_n x^{n+r} \sum_{m=0}^{\infty} q_m x^m = 0
\]

If we collect like powers of \( x \) we obtain

\[
[ a_0 \ell (\ell - 1) + a_0 p_0 \ell + a_0 q_0 ] x^\ell + [a_1 (r+1) \ell + a_1 p_0 (r+1) + a_0 p_1 \ell + a_0 q_1 + a_0 q_0 ] x^{\ell+1} + \ldots + \left[ a_{n-1} (n+r)(n+r-1) + a_n p_0 (n+r) + a_n q_0 + \sum_{m=0}^{n-1} a_{m}(p_{m-(n+r)} + q_{n-m}) \right] x^{n+r} + \ldots = 0
\]

Since \( a_0 \neq 0 \) we get from the first term our indicial equs

\[
F(r) = r (r-1) + p_0 r + q_0 = 0
\]
The roots of the indicial eq. tell us what form our assumed solutions will take on — and how they will behave near the singular point.

Given \( F(r) = r(r-1) + p_0 r + q_0 \), we notice that the coeff. of \( x^{n+r} \) in the series is

\[
A_n F(n+r) + \sum_{m=0}^{n-1} A_m \left[ P_{n-m}(m+r) + q_{n-m} \right] \quad n = 0, 1, \ldots
\]

which must be set to zero to find the recurrence relation,

\[
A_n = \frac{-1}{F(n+r)} \sum_{m=0}^{n-1} A_m \left[ P_{n-m}(m+r) + q_{n-m} \right]. \quad (\ast \ast \ast)
\]

If \( F(r) = 0 \) has the real roots \( r_1 \) and \( r_2 \), \( r_1 \neq r_2 \) then we find

\[
y_1 = |x|^{r_1} \sum_{n=0}^{\infty} A_n(r_1) x^n = |x|^{r_1} \left[ A_0 + \sum_{n=0}^{\infty} A_n(r_1) x^n \right]
\]

where \( A_n(r_1) \) denotes the \( A_n \) value computed from \((\ast \ast \ast)\) using \( r = r_1 \).

Finding \( y_2 \) may be more difficult since if \( r_1 = r_2 + N \) for some positive integer, then in computing \( A_n \) from \((\ast \ast \ast)\) we find that \( n + r_1 \) eventually becomes \( N + r_2 \) which is a root of \( F(r) \) so \( F(N+r_2) = F(r_1) = 0 \).
If either $r_1 = r_2$ or $r_1 = r_2 + 2\pi n$ then in general the second solution $y_2$ will have a logarithmic term.

If $r_1$ and $r_2$ are complex we do not have this difficulty.

Thm 5.7.1 also states that the radius of convergence for all the series solutions found (either one or two) is no smaller than the smaller of the radii of convergences for the two series of

$$x p(x) \quad \text{and} \quad x^2 q(x).$$

**Ex.** #15 **Hint**

$$x(1-x)y'' + \left[ y - (1+\alpha+\beta)x \right]y' - \alpha\beta y = 0$$

\begin{align*}
a) \quad \lim_{x \to 0} \frac{x \left[ y - (1+\alpha+\beta)x \right]}{x(1-x)} &= \gamma \\
\lim_{x \to 0} \frac{x^2 \left[ -\alpha\beta \right]}{x(1-x)} &= 0 \\
\text{and} \quad x(1-x) &= 0 \quad \therefore x = 0
\end{align*}

Indicial eq. $y$ \quad $r(r-1) + \alpha r + \beta = 0$ \quad \text{roots are} \quad r = 0 \quad \text{or} \quad r = 1 - \gamma$

\begin{align*}
b) \quad t &= 1-x \quad u(t) = y(x) \\
1-t &= \frac{du}{dt} \quad u(t) = \frac{d}{dx} u(1-x) = -u'(1-x) = -u'(t) \\
y''(x) &= \frac{d}{dx} (-u'(t)) = u''(t)
\end{align*}

so eq. is

$$t(1-t)u'' - \left[ y - (1+\alpha+\beta)(1-t) \right]u' - \alpha\beta u = 0$$
\[
\begin{align*}
(1-t)u'' + \left[ 1+\alpha+\beta - \gamma + (1+\alpha+\beta) t \right] u' - d\beta u &= 0 \\
\end{align*}
\]

\[
\begin{align*}
t = 0, \quad 0 < \sin \rho \quad \Rightarrow \\
\lim_{t \to 0} t \frac{1+\alpha+\beta - \gamma + (1+\alpha+\beta) t}{t(1-t)} &= 1+\alpha+\beta - \gamma \\
\lim_{t \to 0} t \frac{-d\beta}{t(1-t)} &= 0 \\
\Rightarrow \quad t = 0 \quad \text{is a regular singular point} \quad \Rightarrow \quad x = 1 \quad 0 < \text{regular singular point}.
\end{align*}
\]

Indicial Eq.

\[
\begin{align*}
r(r-1) + \left[ 1+\alpha+\beta - \gamma \right] r &= 0 \\
r^2 - r + r + (\alpha+\beta - \gamma) r &= 0 \\
r (r + \alpha + \beta - \gamma) &= 0 \\
\Rightarrow \quad \begin{cases} 
  r = 0 \\
  r = \gamma - \alpha - \beta 
\end{cases}
\end{align*}
\]

C) \(1-\gamma \) not positive int, and \( a_n = 1 \)

\[
\begin{align*}
y_1 &= |x|^\beta \left[ 1 + \sum_{n=1}^{\infty} a_n x^n \right] = 1 + \sum_{n=1}^{\infty} a_n(x) x^n \\
x P(x) &= \frac{\gamma - (1+\alpha+\beta) x}{1-x} - (1+\alpha+\beta) \frac{x}{1-x} = \frac{\gamma}{1-x} - (1+\alpha+\beta) (1 + \frac{1}{1-x}) \\
&= 1+\alpha+\beta + (\gamma - (1+\alpha+\beta)) \frac{1}{1-x} \quad \text{(geometric series)} \\
&= 1+\alpha+\beta + (\gamma - (1+\alpha+\beta)) (x + x^2 + x^3 + \cdots) \\
&= \gamma + (\gamma - (1+\alpha+\beta)) (x + x^2 + x^3 + \cdots) \\
x^2 P(x) &= -d\beta x \frac{x^2}{x(1-x)} = \frac{x^2}{1-x} = -d\beta x (1+x+x^2 + \cdots) \\
&= -d\beta \left[ x + x^2 + x^3 + \cdots \right]
\end{align*}
\]
So if $y=0$ then from (***) (when $a_0 = 1$)

$$a_1 = \frac{-1}{F(1)} \left[ a_0 \left( p_1 \cdot 0 + q_1 \right) \right] = \frac{-1}{\gamma} \left( 0 + (-2\phi) \right) = \frac{-2\phi}{\gamma}$$

$$a_2 = \frac{-1}{F(2)} \left[ a_0 \left( p_2 \cdot 0 + q_2 \right) + a_1 \left( p_1 \cdot 1 + q_1 \right) \right] = \frac{-1}{2(1+\gamma)} \left[ (-2\phi) + \frac{\phi^2}{\gamma} (2+1-2\phi) \right]$$

$$= \frac{(2\phi)^2}{2\gamma(1+\gamma)}$$