7.4 Basic Theory of Systems of 1st Order Linear Equations

Our general eq. is \( \mathbf{x}' = P\mathbf{x} + \mathbf{g} \)

where \( \mathbf{x}' = \mathbf{x}(t) \)
\( P = P(t) \)
\( \mathbf{g} = \mathbf{g}(t) \)

As in the single eq. case, we will begin by considering the case when \( \mathbf{g}(t) = \mathbf{0} \).

\[ \mathbf{x}' = P\mathbf{x} \quad \text{(\( \ast \))} \]

**Thm 7.4.1** If \( \mathbf{x}^{(n)} \) and \( \mathbf{x}^{(m)} \) are solutions of \( (\ast) \) then \( c_1 \mathbf{x}^{(n)} + c_2 \mathbf{x}^{(m)} \) is also a solution \( \forall c_1, c_2 \).

**Thm 7.4.2** If \( \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)} \) are linearly independent solutions of \( (\ast) \) for each point in \( \alpha < t < \beta \) then each solution \( \mathbf{x} = \mathbf{\phi}(t) \) of \( (\ast) \) can be expressed as

\[ \mathbf{\phi}(t) = c_1 \mathbf{x}^{(1)} + c_2 \mathbf{x}^{(2)} + \cdots + c_m \mathbf{x}^{(m)} \]

in exactly one way.

**Thm 7.4.3** If \( \mathbf{x}^{(n)}, \ldots, \mathbf{x}^{(m)} \) are solutions of \( (\ast) \) on \( \alpha < t < \beta \)

then in this interval \( W[\mathbf{x}^{(n)}, \ldots, \mathbf{x}^{(m)}] \) is either identically zero or else never vanishes.
#7 \( \begin{bmatrix} x^{(1)}(t) \\ x^{(2)}(t) \end{bmatrix} = \begin{bmatrix} t^2 \\ 2t \end{bmatrix} \) \( x^{(3)}(t) = \begin{bmatrix} e^t \\ e^t \end{bmatrix} \)

a) Compute the Wronskian of \( x^{(1)} \) and \( x^{(2)} \)

\[
W = \begin{vmatrix} t^2 & e^t \\ 2t & e^t \end{vmatrix} = t^2 e^t - 2t e^t = t e^t (t - 2)
\]

b) These solutions are linearly independent on \( t < 0 \), \( 0 < t < 2 \), \( 2 < t \)

Since at \( t = 0 \) and at \( t = 2 \), the Wronskian is zero.

c) What conclusion can be drawn about the coefficients in the system of homogeneous differential equations satisfied by \( \dot{x}^{(1)} \) and \( \dot{x}^{(2)} \)?

One or more coefficients must be discontinuous at \( t = 0 \) and at \( t = 2 \).

d) Need to find \( A \) so that \( \dot{x} = A x \) for all \( x = c_1 \dot{x}^{(1)}(t) + c_2 \dot{x}^{(2)}(t) \).

\[
\begin{bmatrix} 2t & e^t \\ 2 & e^t \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e^t \\ e^t \end{bmatrix}
\]

\[
a t^2 + b (2t) = 2t \\
(a + b) e^t = e^t
\]

\[
c t^2 + d (2t) = 2t \\
(c + d) e^t = e^t
\]

\[
a t + 2b = 2 \\
0 + b = 1
\]

\[
c t + 2d = 2 t \\
c + d = 1
\]
\[ \begin{align*}
\text{#7 (continued)} \\
\alpha t + 2b = 2 & \quad a + b = 1 \\
\alpha t + 2(1-a) = 2 & \\
\alpha t + 2 - 2a = 2 & \\
a(t-2) = 0 & \quad a = 0, \quad b = 1 \\
Ct + 2d = \frac{2}{t} & \quad C + d = 1 \\
Ct + 2(1-c) = \frac{2}{t} & \\
Ct + 2 - 2c = \frac{2}{t} & \\
c(t-2) = \frac{2}{t^2} - 2 & \\
c = \frac{2 - 2t}{t(t-2)} & \quad c = \frac{2}{t} \frac{1-t}{t(t-2)} \\
d = \frac{1 - 2 \frac{1-t}{t(t-2)}}{t(t-2) - 2 + 2t}
& = \frac{t^2 - 2}{t(t-2)}
\end{align*} \]

\[ A = \begin{bmatrix}
0 & 1 \\
\frac{2-2t}{t(t-2)} & \frac{t^2 - 2}{t(t-2)}
\end{bmatrix} \]