Predicates and Quantifiers

To be a proposition a statement must be either true or false. Thus statements like \( x > 0 \) are not propositions. This is because their truth value cannot be determined unless additional information is specified; in this case the value of \( x \).

Statements like \( x > 0 \) and \( 3 + x = 5 \) have two parts: a variable (or variables) and a predicate.

We denote these statements \( P(x) \) where \( P(\ ) \) is the predicate and \( x \) is the variable.

Once the variable (or variables) is assigned a value then the truth value of the statement can be determined.

Let \( P(x,y) \) denote the statement "\( x < 2 + y \)." What is the truth value of \( P(2,1) \)?

\[
\begin{align*}
&x < 2 + y \\
&2 < 2 + 1 \\
&2 < 3 \\
&\text{TRUE}
\end{align*}
\]

\( P(x) \) and \( P(x,y) \) are called propositional functions.

When does a propositional function become a proposition?

1. when all variables are assigned values
2. when the truth of the function can be determined for all values of the variable in the universe of discourse.

The second form above is called quantification. Two types of quantification are important:

1. The universal quantification of \( P(x) \) is the proposition \( \forall x \ P(x) \), which is read "for all \( x \), \( P \) of \( x \) (is true)."
2. The existential quantification of \( P(x) \) is the proposition \( \exists x \ P(x) \), which is read "there exists an \( x \) such that \( P \) of \( x \) (is true)."
1. If \( P(x) \) is \( x^2 > 0 \) and the universe of discourse is the real numbers, determine the truth values of both the universal and existential quantifications.

2. Let \( P(x) \) be the statement "\( x \) spends more than five hours every weekday in class," where the universe of discourse for \( x \) is the set of students. Express each of the following quantifications in English:
   
a. \( \exists x \ P(x) \) (answer)  
b. \( \forall x \ P(x) \) (answer)  
c. \( \exists x \ P'(x) \) (answer)  
d. \( \forall x \ \neg P(x) \) (answer) (another answer)

Let \( T(x,y) \) be the propositional function "\( x \) is at least as tall as \( y.\)"
The universe of discourse consists of three students:

Garth 5' 11"
Erin 5' 6"
Marty 6' 0"

Express in words the following propositions and determine their truth value:

a. \( \forall x \ \forall y \ T(x,y) \) (answer)  
b. \( \forall x \ \exists y \ T(x,y) \) (answer)  
c. \( \exists x \ \forall y \ T(x,y) \) (answer)  
d. \( \exists x \ \exists y \ T(x,y) \) (answer)