Let $A$ and $B$ be sets. A **binary relation from $A$ to $B$** is a subset of $A \times B$.

Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Then the following are all relations from $A$ to $B$.

1. $R = \{(1,a), (2,a), (3,b)\}$
2. $S = \{(1,a), (1,b), (2,a)\}$
3. $T = \{(3,a)\}$
4. $U = \{(2,a), (2,b)\}$

Mathematically, if we want to say that $a$ is related to $b$ in some relation $R$ then we write

$$a \mathrel{R} b$$

A **relation on the set $A$** is a relation from $A$ to $A$.

Consider the relation $R = \{(a,b) \mid a \text{ divides } b\}$ on the set $A = \{1,2,3,4,5,6\}$. $R$ consists of ordered pairs in which the first number divides evenly into the second number.

1. List $R$ *(answer)*
2. Display $R$ graphically *(answer)*
3. Display $R$ in tabular form *(answer)*
A relation $R$ on $A$ is **reflexive** if $(a,a)\in R$ for every $a\in A$.

A relation $R$ on $A$ is **symmetric** if $(b,a)\in R$ whenever $(a,b)\in R$ for every $a,b\in A$.

A relation $R$ on $A$ is **antisymmetric** if $(a,b)\in R$ and $(b,a)\in R$ only if $a=b$ for every $a,b\in A$.

A relation $R$ on $A$ is **transitive** if whenever $(a,b)\in R$ and $(b,c)\in R$ then $(a,c)\in R$ for every $a,b,c\in A$.

Since relations are sets, we can combine relations using set operators.

Given two relations $Q$ and $R$ from $A$ to $B$, each of the following operations results in a new relation from $A$ to $B$:

$$Q\cap R, \quad Q\cup R, \quad Q-R, \quad R-Q$$

Let $R$ be a relation from $A$ to $B$ and $S$ be a relation from $B$ to $C$. The **composite** of $R$ and $S$ is the relation consisting of all elements $(a,c)$ where

$$a\in A, b\in B, (a,b)\in R$$
$$b\in B, c\in C, (b,c)\in S$$
$$a\in A, c\in C, (a,c)\in R\circ S$$

Consider the following relations on $\{1,2,3,4\}$. Determine which ones are reflexive, symmetric, antisymmetric or transitive.

$$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$$

$$R_2 = \{(1,1), (1,2), (2,1)\}$$

$$R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}$$

$$R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$$

$$R_5 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$$

$$R_6 = \{(3,4)\}$$

As already seen, we can represent relations several different ways. Consider the relation

$$R = \{(1,1), (1,2), (2,3), (3,3)\}$$

defined on the set $A = \{1, 2, 3, 4\}$

We can construct a table representing this relation. Unlike previous examples, however, now we’ll use zeros and ones to fill the table: a one indicates membership in the relation.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
It’s a small step from the table to a matrix. We’ll call the matrix $M_R$, the matrix representing the relation $R$.

$$M_R = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The $ij$ entry of the $M_R$ matrix is given by

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

Note

- The sets $A$ and $B$ must be in some particular, but arbitrary, order.
- Matrix rows are associated with elements in $A$ and columns are associated with elements in $B$.
- The matrix responding to a relation on a single set $A$ is square.

If $A=\{1,2,3,4\}$ and $B=\{2,4\}$, write the relation that has the matrix $M = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$.

What can be said about the matrix for a relation if the relation is

- reflexive (answer)
- symmetric (answer)
- antisymmetric (answer)

A directed graph or digraph, consists of a set $V$ of vertices (nodes) and a set $E$ of edges (arcs) that point from a particular vertex to a particular vertex.

Let $A = \{a, b, c, d\}$ and let $R = \{(a,a), (a,b), (a,d), (b,d), (c,a), (c,c), (d,d)\}$. Draw the corresponding directed graph.

There is exactly 1 edge for each ordered pair, and the direction of the edge is determined by the order of the pair.
Relations

Digraphs give immediate visual indication of the properties of relations.

**Reflexive**
- Each vertex as an edge looping back to itself

**Symmetric**
- If an edge exists from one vertex to another, then another edge exists from the second vertex back to the first.

**Antisymmetric**
- There are no "symmetric" conditions.

**Transitive**
- If an edge exists from vertex $a$ to vertex $b$ and another edge from vertex $b$ to vertex $c$ then an edge exists from vertex $a$ to vertex $c$.