2.2 The Inverse of a Matrix

If \( A \) is \( n \times n \) (square) and a matrix can be found such that
\[
A \mathbf{x} = \mathbf{I}A = \mathbf{I}
\]
we call \( \mathbf{x} \) the inverse of \( A \) (or \( A \) inverse) and denote it \( A^{-1} \).

Q: How many inverses does a matrix have?

Assume \( B \) is any inverse of \( A \). Then
\[
BA = I \quad \text{and} \quad BAA^{-1} = IA^{-1} = A^{-1}B
\]

So \( A^{-1} \) is unique.

The inverse is very useful. Consider \( A \mathbf{x} = \mathbf{b} \). If \( A \)
is an \( n \times n \) invertible matrix then
\[
A \mathbf{x} = \mathbf{b} \quad \Rightarrow \quad A^{-1}A \mathbf{x} = A^{-1}\mathbf{b} \quad \Rightarrow \quad \mathbf{x} = A^{-1}\mathbf{b}
\]

So (in theory) we can solve the linear system \( A\mathbf{x} = \mathbf{b} \) by finding \( A^{-1} \) and then multiplying \( \mathbf{b} \) on the left by \( A^{-1} \).

Problem: While many square matrices do have inverses, the work necessary to compute them is much greater than solving \( A\mathbf{x} = \mathbf{b} \) without using inverses.
That said, however, we will see that inverses play an important role and are quite useful.

For example, if $A$ (nxn) is invertible so that $x = A^{-1}b$ for any $b \in \mathbb{R}^n$. This tells us that $Ax = b$ is consistent and that it has a unique solution.

Thm

If $A$ is an invertible $nxn$ matrix, then for each $b \in \mathbb{R}^n$, the equation $Ax = b$ has the unique solution $x = A^{-1}b$.

Thm

A) If $A$ is invertible, then $A^{-1}$ is invertible and $(A^{-1})^{-1} = A$.

proof: If $x = A^{-1}$ then $xA = Ax = I$. But this says that $I$ is an invertible matrix whose inverse is $A$. So $(A^{-1})^{-1} = A$.

B) If $A$ and $B$ are invertible $nxn$ matrices, so is $AB$. Also, $(AB)^{-1} = B^{-1}A^{-1}$.

proof: Since $A$ and $B$ are invertible, we have:

- $B^{-1}A^{-1}AB = AB(B^{-1}A^{-1}) = I$
- $(B^{-1}A^{-1})(AB) = (AB)(B^{-1}A^{-1}) = I$

So $(AB)$ is invertible and its inverse $(AB)^{-1}$ is $B^{-1}A^{-1}$.

C) If $A$ is invertible, then so is $A^T$ and $(A^{-1})^T = (A^T)^{-1}$

proof: $AA^T = A^TA = I$

$(AA^T)^T = (A^TA)^T = I^T = I$

$(A^{-1})^T = (A^T)^{-1}$.

so the inverse of $A^T$ is $(A^{-1})^T$. Sometimes $(A^{-1})^T$ is denoted $A^{-T}$.
Q: Given $A$, how do we find $A^{-1}$ (if it exists)?

Assuming $A^{-1}$ exists, set $B = A^{-1}$ so that $AB = I$.

$$AB = A [\vec{b}_1 \ b_2 \ \ldots \ b_n] = [A\vec{b}_1 \ A\vec{b}_2 \ \ldots \ A\vec{b}_n] \quad \text{but} \quad AB = I = [\vec{e}_1 \ \vec{e}_2 \ \ldots \ \vec{e}_n]$$

so we get $n$ linear systems $A\vec{b}_i = \vec{e}_i$.

$$A\vec{b}_1 = \vec{e}_1$$
$$A\vec{b}_2 = \vec{e}_2$$
$$A\vec{b}_n = \vec{e}_n$$

Each of these can be solved by row reducing $[A \ \vec{e}_i]$ to get $\vec{b}_i$.

I.e. row reduce

$$[\vec{a}_1 \ \vec{a}_2 \ \ldots \ \vec{a}_n \ \vec{e}_1]$$

$$\vdots$$

$$[\vec{a}_1 \ \vec{a}_2 \ \ldots \ \vec{a}_n \ \vec{e}_n]$$

But we can do this all at once by row reducing

$$[\vec{a}_1 \ \vec{a}_2 \ \ldots \ \vec{a}_n \ \vec{e}_1 \ \vec{e}_2 \ \ldots \ \vec{e}_n]$$

or

$$[A \ I]$$

Since if $A$ is invertible it has $n$-pivot columns (why?) we know $A$ is row equivalent to $I$. When we row reduce $[A \ I]$ we end up with $[I \ B']$, but recall that $B = A^{-1}$.

Thm An $n \times n$ matrix $A$ is invertible iff it is row equivalent to $I_n$. 
Proof: Let $E$ be an elementary matrix, obtained by performing a single elementary row operation on $I_n$. Note that $EA$ yields a matrix identical to performing the same row operation on $A$ itself.

If $A$ is row equivalent to $I_n$, then there exist elementary matrices $E_1, E_2, \ldots, E_p$ such that

$$E_p E_{p-1} \cdots E_2 E_1 A = I_n$$

In this case $A^{-1} = E_p E_{p-1} \cdots E_2 E_1$, so $A^{-1}$ exists.

If $A$ is invertible then $A^{-1}$ exists such that $A^{-1} A = I_n$.

Let $E_1, E_2, \ldots, E_p$ be elementary matrices such that

$$E_p \cdots E_1 I_n = A^{-1}.$$ These are the same operations that turn $A$ into $I_n$, since $E_p \cdots E_1 A = A^{-1} A = I_n$.

Therefore $A$ is row equivalent to $I_n$. 

Ex Find the inverse of $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$.

\[
\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & -\frac{1}{2} \end{bmatrix} 
\]

\[
\begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} 
\]

So $A^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

Check $AA^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 4 - 3 & -2 + 2 \\ 6 - 6 & -3 + 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \checkmark$

$A^{-1}A = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 4 - 3 & 2 - 2 \\ -6 + 6 & -3 + 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \checkmark$

For the $2 \times 2$ matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we know how to compute the inverse (and check for existence).

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

which only exists if $ad - bc \neq 0$.

Text discusses elementary matrices, which are obtained by performing a single row operation on $I$. If $E$ is such a matrix the $EA$ is that matrix which would result from performing the same row operation on $A$ itself.

Discuss: 2.2: 16, 18, 22

Octave uses $\text{inv}(A)$ to compute the inverse of $A$. Sage uses $A\text{.inverse()}$. 