2.3 Characterization of Invertible Matrices

Thm 8 Let $A$ be a square $n \times n$ matrix. The following statements are equivalent:

a) $A$ is an invertible matrix
b) $A$ is row equivalent to $I$
c) $A$ has $n$ pivot positions
d) The equation $AX = 0$ has only the trivial solution
e) The columns of $A$ form a linearly independent set
f) The linear transformation $x \mapsto Ax$ is one-to-one.
g) The equation $AX = I$ has at least one solution for each $b \in \mathbb{R}^n$

[I can also be stated as "$AX = b$ has a unique solution for each $b \in \mathbb{R}^n$."

h) The columns of $A$ span $\mathbb{R}^n$
i) The linear transformation $x \mapsto AX$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$
j) $A$ is an $n \times n$ matrix $C$ s.t. $CA = I$
k) $A$ is an $n \times n$ matrix $D$ s.t. $AD = I$
l) $A^T$ is an invertible matrix.

The set of $n \times n$ matrices is partitioned into two disjoint sets:

invertible (nonsingular) non-invertible (singular)
A linear transformation \( T : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is invertible if there exists \( S : \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that:

\[
S(T(x)) = x \quad \forall x \in \mathbb{R}^n \\
T(S(x)) = x \quad \forall x \in \mathbb{R}^n
\]

**Thm 9**

Let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a linear transformation and let \( A \) be the standard matrix of that transformation. Then \( T \) is invertible iff \( A \) is an invertible matrix.

**Ex 11.** a) \( T \) d=6 b) \( T \) h=|c| c) \( T \) d d) \( T \) c)

**Ex 13.** \[
\begin{pmatrix}
a & x & x_1 \\
o & b & x_2 \\
o & o & c
\end{pmatrix}
\]

If \( a, b, c \neq 0 \) we can row reduce this to \( I_3 \).

What if \( a, b \) or \( c = 0 \)?

- \( a = 0 \) \( \Rightarrow \) column of zeros, cannot row reduce to \( I_3 \)
- \( b = 0 \) \( \Rightarrow \) only 2 pivot columns
- \( c = 0 \) \( \Rightarrow \) ditto

A triangular \( n \times n \) matrix is invertible iff each of its diagonal entries is nonzero.