Consider the $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. How do row operations on $A$ affect the determinant?

1. **Row replacement:** $kR_1 + R_2 \to R_2$ gives $\begin{bmatrix} a & b \\ c+ka & d+kb \end{bmatrix}$

\[
\det \begin{bmatrix} a & b \\ c+ka & d+kb \end{bmatrix} = ad + kab - bc - kab \\
= ad - bc \\
= \det A
\]

This row replacement did not change the value of the determinant. In fact, this is the case for any row replacement on any size matrix.

\[
\text{Row Replacement operation do not change the determinant of a matrix.}
\]

2. **Row scaling:** $kR_1 \to R_1$ gives $\begin{bmatrix} ka & kb \\ c & d \end{bmatrix}$

\[
\det \begin{bmatrix} ka & kb \\ c & d \end{bmatrix} = kad - kbc \\
= k(ad - bc) \\
= k \det A
\]

\[
\text{Scaling a row by a factor } k \text{ results in a matrix whose determinant is } k \text{ times the determinant of the original matrix.}
\]
3.2 Properties of Determinants

3. Row Swapping: \( R_1 \leftrightarrow R_2 \)

\[
\begin{vmatrix} c & d \\ a & b \end{vmatrix}
\]

\[
\det \begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad \\
= -(ad - cb) \\
= - \det A
\]

Swapping two rows of a matrix changes the sign of the determinant.

Recall that the determinant of a triangular matrix is the product of its diagonal elements. Idea: We should be able to perform row replacement and row swapping operations on any square matrix to obtain a triangular matrix, and because row replacement does not change the determinant and row swapping only changes the sign of the determinant, we can compute the determinant of a square matrix by finding the corresponding triangular matrix, find its determinant, and multiply this by \((-1)^r\) where \( r \) is the number of row swaps.

If \( A \sim U \) and \( U \) is obtained from \( A \) by row replacements and \( r \) row swaps, then

\[
\det A = (-1)^r \det U
\]
3.2 Properties of Determinants

\[ \text{Ex} \quad A = \begin{bmatrix} 3 & -6 & 3 \\ 6 & -12 & 2 \\ -1 & 7 & 0 \end{bmatrix} \]

\[ A \sim \begin{bmatrix} 3 & -6 & 3 \\ 0 & 0 & -4 \\ 0 & 5 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & -6 & 3 \\ 0 & 5 & 1 \\ 0 & 0 & -4 \end{bmatrix} = U \]

\[ \det(U) = (3)(5)(-4) = -60 \]

1 row exchange so

\[ \det(A) = (-1)^1 (-60) = 60 \]

\[ \det(A) = 60 \]

Note: If \( A = BU \) then

1) \( |\det(A)| = |\det(U)| \)

2) \( \det(A) = 0 \) iff \( \det(U) = 0 \)

3) \( \det(U) = 0 \) iff there is a zero on the diagonal

4) \( U \) is invertible iff it has \( n \) pivots

5) \( U \) is invertible iff it has no zeros on the diagonal

6) \( A \sim I \) iff \( U \sim I \)

7) \( A \) is invertible iff \( U \) is invertible

\[ \therefore \text{The n} \times \text{n matrix } A \text{ is invertible iff } \det(A) \neq 0 \]
3.2 Properties of Determinants

Some other interesting properties

Q. What is the relation between $\det A$ and $\det kA$?

A. In the matrix $kA$ each row of $A$ has been scaled by $k$. Since scaling one row of $A$ changes the determinant by a factor of $k$, scaling $n$ rows of $A$ will change the determinant by a factor of $k^n$.

\[
\det kA = k^n \det A \text{ if } A \text{ is } nxn
\]

Q. What is $\det A^T$?

A. Suppose we compute $\det A$ using cofactor expansion about row $i$. The identical computation will be performed when computing $\det A^T$ using column $i$. Thus

\[
\det A = \det A^T
\]

Distributive Property - $\det(AB) = \det A \cdot \det B$

Proof: If $\det A = 0$ then $A$ is not invertible. This tells us that $AB$ is not invertible. (If $AB$ is invertible then $A$ is a $nxn$ matrix $W$ s.t. $ABW = I$, but then $A(BW) = I$ so $A$ is invertible). If $AB$ is not invertible then $\det AB = 0$ and our statement is true.
3.2 Properties of Determinants

If \( A \) is invertible then \( A \sim I \) and so there exist elementary matrices \( E_p, E_{p-1}, \ldots, E_1 \) such that
\[
A = E_p E_{p-1} \ldots E_1 \cdot I = E_p E_{p-1} \ldots E_1.
\]

\[
\text{det}(AB) = \text{det}(E_p E_{p-1} \ldots E_1, B)
\]
\[
= \text{det}(E_p) \text{det}(E_{p-1} \ldots E_1, B)
\]

because \( \text{det}(E_p) \) will be 1, -1, or \( k \)

- 1 if \( E_p \) is a row replacement
- 1 if \( E_p \) is a row swap
- \( k \) if \( E_p \) scales a row by \( k \)

Continuing we have

\[
\text{det}(AB) = \text{det}(E_p) \text{det}(E_{p-1}) \ldots \text{det}(E_1) \text{det}(B)
\]
\[
= \text{det}(E_p E_{p-1} \ldots E_1) \text{det}(B)
\]
\[
= \text{det} A \cdot \text{det} B.
\]