5.6 Dynamical Systems

Consider \( A = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.7 \end{bmatrix} \) We know the eigenvalues are 
\( \lambda_1 = 0.9 \quad \lambda_2 = 0.7 \)

We can also find the eigenvectors - they are 
\[ \vec{v}_1 = \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

If this matrix were part of an iteration \( \vec{x}_{k+1} = A \vec{x}_k \) 
we can use the eigenvalues and eigenvectors to study its behavior.

To carry out the iteration we must begin with an initial vector \( \vec{x}_0 \). Express this in terms of \( \vec{v}_1 \) and \( \vec{v}_2 \)

\[ \vec{x}_0 = c_1 \vec{v}_1 + c_2 \vec{v}_2 \] (ask why this is always possible)

Now, compute \( \vec{x}_1, \vec{x}_2, \vec{x}_3, \ldots \)

\[ \vec{x}_1 = A \vec{x}_0 = A \left[ c_1 \vec{v}_1 + c_2 \vec{v}_2 \right] = c_1 A \vec{v}_1 + c_2 A \vec{v}_2 = c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2 \]
\[ \vec{x}_2 = A \vec{x}_1 = A \left[ c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2 \right] = c_1 \lambda_1^2 \vec{v}_1 + c_2 \lambda_2^2 \vec{v}_2 \]

\[ \vec{x}_k = A \vec{x}_{k-1} = c_1 \lambda_1^k \vec{v}_1 + c_2 \lambda_2^k \vec{v}_2 \]

Thus, the value of any \( \vec{x}_k \) can be computed easily 
by raising \( \lambda_1 \) and \( \lambda_2 \) to the \( k \)-th power and 
doing a linear combination.
For our example, we find \( \bar{X}_k = C_1 (1.9)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 (1.7)^k \begin{bmatrix} 0 \\ 1 \end{bmatrix} \).

**Q:** What does \( \bar{X}_k \) approach as \( k \to \infty \)?

Since \( (1.9)^k \to 0 \) and \( (1.7)^k \to 0 \) we expect \( \bar{X}_k \to \bar{0} \). This happens regardless of any \( \bar{X}_0 = [\bar{X}_1 \bar{X}_2] \) vector.

In fact, since \( (1.1)^k \to 0 \) faster than \( (1.9)^k \to 0 \), we expect that the second coordinate of \( \bar{X}_k \) will approach zero more rapidly than the first as \( k \to \infty \).

<table>
<thead>
<tr>
<th>( k )</th>
<th>( (1.9)^k )</th>
<th>( (1.7)^k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>.9</td>
<td>.7</td>
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<tr>
<td>2</td>
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<tr>
<td>3</td>
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<td>.24</td>
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<td>6</td>
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<td>7</td>
<td>.48</td>
<td>.08</td>
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<tr>
<td>8</td>
<td>.43</td>
<td>.06</td>
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<tr>
<td>9</td>
<td>.39</td>
<td>.04</td>
</tr>
</tbody>
</table>

In this case, the origin is an **attractor**. No matter where one starts, as \( k \to \infty \) you end up at the origin.

Notice that we always approach the origin (in this example) along the \( x_1 \) axis (\( \bar{V}_1 \) eigenvector) unless we start on the \( x_2 \) axis (along \( \bar{V}_2 \)).

This is because \( \lambda_2 = 0.7 < \lambda_1 = 0.9 \).
Now consider \( A = \begin{bmatrix} 1.5 & 0 \\ 0 & 2 \end{bmatrix} \) \( \lambda_1 = 1.5 \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \)
\( \lambda_2 = 2 \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \)

\( \vec{x}_{k+1} = A \vec{x}_k \) with \( \vec{x}_0 = c_1 \vec{v}_1 + c_2 \vec{v}_2 \) gives us again:

\[
\vec{x}_k = c_1 \lambda_1^k \vec{v}_1 + c_2 \lambda_2^k \vec{v}_2
\]

\[
\vec{x}_k = c_1 (1.5)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 (2)^k \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 (1.5)^k \\ c_2 2^k \end{bmatrix}
\]

Now the origin is a repeller since for any \( \vec{x}_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \) we are pushed away from the origin as \( k \) increases.

Since (in this example) the \( x_1 \) coordinate increases the fastest, we increasingly move in that direction as \( k \to \infty \).

If \( |\lambda| < 1 \) then
we will be attracted toward
the origin in the \( \vec{v}_1 \) direction

If \( |\lambda| > 1 \) then we are repelled from
the origin in the \( \vec{v}_1 \) direction

Suppose we mixed these two...
\[ A = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix} \]

\[ \lambda_1 = 2 \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

\[ \lambda_2 = 0.5 \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

We expect to move toward the origin along \( \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) but move away from the origin along \( \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \)

Here again \( \vec{x}_k = C_1 \lambda_1^k \vec{v}_1 + C_2 \lambda_2^k \vec{v}_2 \)

\[ \vec{x}_1 = C_1 (2)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 (0.5)^k \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} C_1 2^k \\ C_2 (0.5)^k \end{bmatrix} \]

Now the origin is a **saddle point**.

If \( C_1 = 0 \) then we start on the \( x_2 \) axis and we will be attracted to the origin. However, if we start even a little off the \( x_2 \) axis we end up moving away from the origin with the \( x_1 \) axis as an asymptote.

In fact, for large \( k \) \( \vec{x}_k \approx 2^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} \)

All of this work has been with diagonal matrices. If the matrix is not diagonal then the outcome will be similar except that the eigenvectors \( \vec{v}_1 \) and \( \vec{v}_2 \) will not (in general) point along the axes.
\[ A = \begin{bmatrix} 1.25 & 0.75 \\ 0.75 & 1.25 \end{bmatrix} \]
\[ \lambda_1 = 2 \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]
\[ \lambda_2 = 0.5 \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \]

\[
\begin{aligned}
\vec{x}_{k+1} &= A \vec{x}_k \\
\vec{x}_0 &= \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}
\end{aligned}
\]
\[
\begin{aligned}
\vec{x}_k &= c_1 \lambda_1^k \vec{v}_1 + c_2 \lambda_2^k \vec{v}_2 \\
\vec{x}_k &= c_1 (2)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 (0.5)^k \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\end{aligned}
\]

Some behavior as last time, just different directions.

Finally, consider \[ A = \begin{bmatrix} 2.3 & -2.5 \\ 1 & -0.7 \end{bmatrix} \]
\[ \lambda_1 = 0.8 + 0.5i \quad \vec{v}_1 = \begin{bmatrix} 0.6 - 0.2i \\ 1 \end{bmatrix} \]
\[ \lambda_2 = 0.8 - 0.5i \quad \vec{v}_2 = \begin{bmatrix} 0.6 + 0.2i \\ 1 \end{bmatrix} \]

Complex eigenvalues and eigenvectors lead to solutions that spiral about the origin.

If \( \text{Re} \lambda < 0 \) \implies spiral in to origin.
If \( \text{Re} \lambda > 0 \) \implies spiral out away from origin.
If \( \text{Re} \lambda = 0 \) \implies rotate around origin in an ellipse.