COMPUTING AND ESTIMATING THE RATE OF CONVERGENCE

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Abstract. Introduces the definition of rate of convergence for sequences and applies this to fixed-point root-finding iterative methods. Concludes with the development of a formula to estimate the rate of convergence for these methods when the actual root is not known.

1. Rate of Convergence

Definition 1. If a sequence \( x_1, x_2, \ldots, x_n \) converges to a value \( r \) and if there exist real numbers \( \lambda > 0 \) and \( \alpha \geq 1 \) such that

\[
\lim_{n \to \infty} \frac{|x_{n+1} - r|}{|x_n - r|^\alpha} = \lambda
\]

then we say that \( \alpha \) is the rate of convergence of the sequence.

When \( \alpha = 1 \) we say the sequence converges linearly and when \( \alpha = 2 \) we say the sequence converges quadratically. If \( 1 < \alpha < 2 \) then the sequence exhibits superlinear convergence.

2. Fixed-Point Iterations

Many root-finding methods are fixed-point iterations. These iterations have this name because the desired root \( r \) is a fixed-point of a function \( g(x) \), i.e., \( g(r) \to r \).

To be useful for finding roots, a fixed-point iteration should have the property that, for \( x \) in some neighborhood of \( r \), \( g(x) \) is closer to \( r \) than \( x \) is. This leads to the iteration

\[
x_{n+1} = g(x_n), \quad n = 0, 1, 2, \ldots
\]

Newton’s method is an example of a fixed-point iteration since

\[
x_{n+1} = g(x_n), \quad g(x) = x - \frac{f(x)}{f'(x)}
\]

and clearly \( g(r) = r \) since \( f(r) = 0 \).

Theorem 1. Let \( r \) be a fixed-point of the iteration \( x_{n+1} = g(x_n) \) and suppose that \( g'(r) \neq 0 \). Then the iteration will have a linear rate of convergence.

Proof. Using Taylor’s Theorem for an expansion about fixed-point \( r \) we find

\[
g(x) = g(r) + g'(r)(x - r) + \frac{g''(\xi)}{2}(x - r)^2
\]

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where $\xi$ is some value between $x$ and $r$. Evaluating at $x_n$ and noting that $x_{n+1} = g(x_n)$ and $g(r) = r$ we obtain

$$x_{n+1} = r + g'(r)(x_n - r) + \frac{g''(r)}{2}(x_n - r)^2.$$ 

Subtracting $r$ from both sides and dividing by $x_n - r$ gives

$$\frac{x_{n+1} - r}{x_n - r} = g'(r) + \frac{g''(r)}{2}(x_n - r)$$

which, as $n \to \infty$, yields

$$\lim_{n \to \infty} \frac{|x_{n+1} - r|}{|x_n - r|} = |g'(r)|.$$ 

Comparing this with Equation (1) we see that $\alpha = 1$ and $\lambda = |g'(r)|$, indicating that the method converges linearly. $\square$

Next, consider the case when $g'(r) = 0$. This is important because it explains why Newton’s method converges so quickly (when it converges at all).

**Theorem 2.** Let $r$ be a fixed-point of the iteration $x_{n+1} = g(x_n)$ and suppose that $g'(r) = 0$ but $g''(r) \neq 0$. Then the iteration will have a **quadratic** rate of convergence.

**Proof.** Using Taylor’s Theorem once again, but including one more term, we have

$$g(x) = g(r) + g'(r)(x - r) + \frac{g''(r)}{2}(x - r)^2 + \frac{g'''(r)}{6}(x - r)^3.$$ 

As before, we substitute $x_n$ for $x$ and use the facts that $x_{n+1} = g(x_n)$, $g(r) = r$, and $g'(r) = 0$ to obtain

$$x_{n+1} = r + \frac{g''(r)}{2}(x_n - r)^2 + \frac{g'''(r)}{6}(x_n - r)^3.$$ 

Subtracting $r$ from both sides and dividing by $(x_n - r)^2$ gives

$$\frac{x_{n+1} - r}{(x_n - r)^2} = \frac{g''(r)}{2} + \frac{g'''(r)}{6}(x_n - r)$$

which, as $n \to \infty$, gives

$$\lim_{n \to \infty} \frac{|x_{n+1} - r|}{|x_n - r|^2} = \frac{|g''(r)|}{2}.$$ 

Observe that $\alpha = 2$, which shows the iteration will converge quadratically. $\square$

In most instances this situation applies to Newton’s method. Computing $g'(x)$ from (2) we have

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{|f'(x)|^2} = \frac{f(x)f''(x)}{|f'(x)|^2}.$$ 

When this is evaluated at $r$, we find that $g'(r) = 0$ because $f(r) = 0$, provided $f'(r) \neq 0$, and so we expect Newton’s method will converge quadratically. It is possible to show that

$$\lim_{x \to r} g'(x) = \frac{1}{2}$$

when $f'(r) = 0$, so in this case Newton’s method exhibits only linear convergence.
3. Estimating the Rate of Convergence

It is convenient to define the error after \( n \) steps of an iterative root-finding algorithm as \( e_n = x_n - r \). As \( n \to \infty \) we see from Equation (1) that

\[
|e_{n+1}| \approx \lambda |e_n|^\alpha \quad \text{and} \quad |e_n| \approx \lambda |e_{n-1}|^\alpha
\]

and so

\[
\frac{|e_{n+1}|}{|e_n|} \approx \frac{\lambda |e_n|^\alpha}{\lambda |e_{n-1}|^\alpha} \approx \left( \frac{e_n}{e_{n-1}} \right)^\alpha.
\]

Solving for \( \alpha \) yields

\[
(6) \quad \alpha \approx \log \frac{|e_{n+1}/e_n|}{|e_n/e_{n-1}|} = \frac{\log |(x_{n+1} - r)/(x_n - r)|}{\log |(x_n - r)/(x_{n-1} - r)|}.
\]

To make use of this formula we need to know the ratios of consecutive errors. While we cannot compute these ratios exactly (since we do not know the exact value of the root \( r \)), we can approximate them with ratios of the differences of consecutive estimates of the root. To see this, first substitute \( x_n \) and \( x_{n-1} \) into Equation (3) to obtain the two expansions

\[
(7) \quad x_{n+1} = r + g'(r)(x_n - r) + \frac{g''(r)}{2}(x_n - r)^2
\]

\[
(8) \quad x_n = r + g'(r)(x_{n-1} - r) + \frac{g''(r)}{2}(x_{n-1} - r)^2.
\]

Subtracting (8) from (7), dividing by \( (x_n - x_{n-1}) \), and taking the limit as \( n \to \infty \) we have

\[
\lim_{n \to \infty} \frac{x_{n+1} - x_n}{x_n - x_{n-1}} = \lim_{n \to \infty} \left| g'(r) + \frac{g''(r)}{2} \left( \frac{x_n - r}{x_n - x_{n-1}} \right)^2 - \frac{g''(r)}{2} \right| = \| g'(r) \|
\]

since both \( (x_n - r)^2 \) and \( (x_{n-1} - r)^2 \) go to zero more quickly than \( x_n - x_{n-1} \) as the sequence \( \{x_n\} \) converges to \( r \). Comparing this result with Equation (4) we conclude, for suitably large values of \( n \), that

\[
\frac{e_{n+1}}{e_n} = \frac{x_{n+1} - r}{x_n - r} \approx \frac{x_n - r}{x_{n-1} - r} \approx \frac{x_{n+1} - x_n}{x_n - x_{n-1}}
\]

which allows us to approximate \( \alpha \) with

\[
(9) \quad \alpha \approx \frac{\log |(x_{n+1} - x_n)/(x_n - x_{n-1})|}{\log |(x_n - x_{n-1})/(x_{n-1} - x_{n-2})|}.
\]

Even though this only gives an estimate of \( \alpha \), we note that in practice it agrees well with the theoretical convergence rates of bisection and Newton’s method and gives us a good measure of the efficiency of various forms of fixed-point algorithm.