

# GRAPH THEORY: BASIC DEFINITIONS AND THEOREMS

## 1. DEFINITIONS

**Definition 1.** A **graph**  $G = (V, E)$  consists of a set  $V$  of **vertices** (also called **nodes**) and a set  $E$  of **edges**.

**Definition 2.** If an edge connects to a vertex we say the edge is **incident** to the vertex and say the vertex is an **endpoint** of the edge.

**Definition 3.** If an edge has only one endpoint then it is called a **loop edge**.

**Definition 4.** If two or more edges have the same endpoints then they are called **multiple** or **parallel** edges.

**Definition 5.** Two vertices that are joined by an edge are called **adjacent** vertices.

**Definition 6.** A **pendant** vertex is a vertex that is connected to exactly one other vertex by a single edge.

**Definition 7.** A **walk** in a graph is a sequence of alternating vertices and edges  $v_1e_1v_2e_2\ldots v_ne_nv_{n+1}$  with  $n \geq 0$ . If  $v_1 = v_{n+1}$  then the walk is **closed**. The **length** of the walk is the number of edges in the walk. A walk of length zero is a **trivial walk**.

**Definition 8.** A **trail** is a walk with no repeated edges. A **path** is a walk with no repeated vertices. A **circuit** is a closed trail and a **trivial circuit** has a single vertex and no edges. A trail or circuit is **Eulerian** if it uses every edge in the graph.

**Definition 9.** A **cycle** is a nontrivial circuit in which the only repeated vertex is the first/last one.

**Definition 10.** A **simple graph** is a graph with no loop edges or multiple edges. Edges in a simple graph may be specified by a set  $\{v_i, v_j\}$  of the two vertices that the edge makes adjacent. A graph with more than one edge between a pair of vertices is called a **multigraph** while a graph with loop edges is called a **pseudograph**.

**Definition 11.** A **directed graph** is a graph in which the edges may only be traversed in one direction. Edges in a simple directed graph may be specified by an ordered pair  $(v_i, v_j)$  of the two vertices that the edge connects. We say that  $v_i$  is **adjacent to**  $v_j$  and  $v_j$  is **adjacent from**  $v_i$ .

**Definition 12.** The **degree** of a vertex is the number of edges incident to the vertex and is denoted  $\deg(v)$ .

**Definition 13.** In a directed graph, the **in-degree** of a vertex is the number of edges **incident to** the vertex and the **out-degree** of a vertex is the number of edges **incident from** the vertex.

**Definition 14.** A graph is **connected** if there is a walk between every pair of distinct vertices in the graph.

**Definition 15.** A graph  $H$  is a **subgraph** of a graph  $G$  if all vertices and edges in  $H$  are also in  $G$ .

**Definition 16.** A **connected component** of  $G$  is a connected subgraph  $H$  of  $G$  such that no other connected subgraph of  $G$  contains  $H$ .

**Definition 17.** A graph is called **Eulerian** if it contains an Eulerian circuit.

**Definition 18.** A **tree** is a connected, simple graph that has no cycles. Vertices of degree 1 in a tree are called the **leaves** of the tree.

**Definition 19.** Let  $G$  be a simple, connected graph. The subgraph  $T$  is a **spanning tree of**  $G$  if  $T$  is a tree and every node in  $G$  is a node in  $T$ .

**Definition 20.** A **weighted graph** is a graph  $G = (V, E)$  along with a function  $w : E \rightarrow \mathbb{R}$  that associates a numerical weight to each edge. If  $G$  is a weighted graph, then  $T$  is a **minimal spanning tree of**  $G$  if it is a spanning tree and no other spanning tree of  $G$  has smaller total weight.

**Definition 21.** The **complete graph** on  $n$  nodes, denoted  $K_n$ , is the simple graph with nodes  $\{1, \dots, n\}$  and an edge between every pair of distinct nodes.

**Definition 22.** A graph is called **bipartite** if its set of nodes can be partitioned into two disjoint sets  $S_1$  and  $S_2$  so that every edge in the graph has one endpoint in  $S_1$  and one endpoint in  $S_2$ .

**Definition 23.** The **complete bipartite graph** on  $n, m$  nodes, denoted  $K_{n,m}$ , is the simple bipartite graph with nodes  $S_1 = \{a_1, \dots, a_n\}$  and  $S_2 = \{b_1, \dots, b_m\}$  and with edges connecting each node in  $S_1$  to every node in  $S_2$ .

**Definition 24.** Simple graphs  $G$  and  $H$  are called **isomorphic** if there is a bijection  $f$  from the nodes of  $G$  to the nodes of  $H$  such that  $\{v, w\}$  is an edge in  $G$  if and only if  $\{f(v), f(w)\}$  is an edge of  $H$ . The function  $f$  is called an **isomorphism**.

**Definition 25.** A simple, connected graph is called **planar** if there is a way to draw it on a plane so that no edges cross. Such a drawing is called an **embedding** of the graph in the plane.

**Definition 26.** For a planar graph  $G$  embedded in the plane, a **face** of the graph is a region of the plane created by the drawing. The area of the plane outside the graph is also a face, called the unbounded face.

## 2. THEOREMS

**Theorem 1.** *Let  $G$  be a connected graph. Then  $G$  is Eulerian if and only if every vertex in  $G$  has even degree.*

**Theorem 2** (Handshaking Lemma). *In any graph with  $n$  vertices  $v_i$  and  $m$  edges*

$$\sum_{i=1}^n \deg(v_i) = 2m$$

**Corollary 1.** *A connected non-Eulerian graph has an Eulerian trail if and only if it has exactly two vertices of odd degree. The trail begins and ends these two vertices.*

**Theorem 3.** *If  $T$  is a tree with  $n$  edges, then  $T$  has  $n + 1$  vertices.*

**Theorem 4.** *Two graphs that are isomorphic to one another must have*

- (1) *The same number of nodes.*
- (2) *The same number of edges.*
- (3) *The same number of nodes of any given degree.*
- (4) *The same number of cycles.*
- (5) *The same number of cycles of any given size.*

**Theorem 5** (Kuratowski's Theorem). *A graph  $G$  is nonplanar if and only if it contains a "copy" of  $K_{3,3}$  or  $K_5$  as a subgraph.*

**Theorem 6** (Euler's Formula for Planar Graphs). *For any connected planar graph  $G$  embedded in the plane with  $V$  vertices,  $E$  edges, and  $F$  faces, it must be the case that*

$$V + F = E + 2.$$