Contemporary Mathematics

The Borda Count, the Kemeny Rule, and the Permutahedron

Karl-Dieter Crisman

Abstract. When thinking about choice beyond single winners, social preference functions are natural to study; these are functions where both input and output are strict rankings of n items (or possibly ties among several such rankings). Symmetry is one mathematical way to express fairness, so it makes sense to study the symmetry of these functions carefully.

Such rankings may be viewed as a permutation of the items; since pairwise comparison is also important in voting, a natural combinatorial object for studying such functions is the permutahedron. This paper analyzes a large class of social preference functions using the representation theory of the symmetry group of the permutahedron. The main result identifies the most symmetric possible family in this class, which preserves pairwise information fully; it is the one-parameter family that connects the Borda Count and the Kemeny Rule.

1. Introduction

1.1. Why Social Preference Functions? Choice questions are typically about aggregating individual preferences into a 'societal' preference. For example, with n choices, $A_1, A_2, \ldots, A_n$, any individual voter's preference is represented as a strict transitive ranking such as $A_1 \succ A_3 \succ \cdots \succ A_2$; some mathematical rule then yields an aggregate result. Different types of outcomes, whether singletons or choice functions, yield different categories of functions.

In some natural situations the actual outcome should be a ranking, or some related structure. For instance, a group could choose officers (Chair, Secretary, and Treasurer) from three candidates a nominating committee gives them. The offices might have a priority order (like for succession), but their priority status is not the only point. Even more interesting, one might have a list of factories to inspect for an internal audit. Here, the cyclic order of a visiting schedule likely is more important than which factory actually is the first one visited in the year.

In any similar case, it is reasonable to assume that the output of the function is one *or more* strict rankings, just as in a voting function the output is one or more candidates. We call such a function a *social preference function.* The most famous s.p.f. is probably the Kemeny Rule.

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A special class of social preference functions has been receiving some attention in recent work, especially in the two pairs of articles [**23, 29**] and [**34, 35**]. This class, called *simple ranking scoring functions* (SRSFs – see section 2.1), fills exactly the same role in the class of social preference functions as the usual positional scoring rules do in the class of voting rules[1].

The point is that in many situations where choice matters we may wish to consider not just the relative rank of the candidates, but where they fit into the overall order. So one goal of this paper is to introduce an interesting generalization of several well-known procedures of value in contexts beyond single-winner elections.

1.2. Symmetry. Since individual preferences may be represented as combinatorial objects, the symmetries of such objects are often of interest. Though they do not use the same formal language, all the foundational papers (e.g., [**2, 25, 4, 10**]) constantly refer to such symmetry[2]. For instance, one might think of rankings as permutations of the set of candidates $\{1, 2, \dots, n\}$, and seek information about permutations which yields information about different procedures. For a game theory example, coalitions on up/down votes (ignoring abstentions) in the United Nations Security Council are simply subsets of the power set of the set of voters.

Many 'natural' fairness requirements in social choice can be thought of in terms of a natural group action on such combinatorial structures. Put another way, invariance of a procedure under a group action could be considered more equitable.

Indeed, symmetry under the action of the symmetric group on n candidates $\{A_i\}_{i=1}^n$ (denoted throughout by S_n) is usually known as *neutrality*; the idea is that no candidate has an unfair advantage[3]. The same action on the set of voters is usually useful in a game theory context; the Security Council does *not* have this symmetry, due to the five members with veto power.

It is fruitful to study a large range of rules to see how they behave with regard to various symmetries; over time, the field of social choice has moved in this direction from a more axiomatic approach. Structural papers like [**18, 19**] and the social preference function papers cited above are solidly within this tradition; another goal of this paper is to add to this classification literature.

1.3. Context for this work. Within the last few years, representation theory has become a tool to reframe and powerfully extend previous classifications. Orrison and his students [**7**] have done so in voting theory, while work of Hernández-Lamoneda, Juárez, and Sánchez-Sánchez [**12**] gives similar results in cooperative game theory. These techniques are also used in work of Bargagliotti and Orrison in nonparametric statistics [**3**].

In these papers, representations of S_n allow generalization with fewer technical challenges, with more insight into *why* the results are true. But there is more to combinatorics than permutations, and more to fairness than the symmetric group. Pairwise comparisons between candidates have been a cornerstone of voting theory

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[1] We note that [**15**] and some recent preprints by Pivato and Nehring address an even *more* general group of functions.

[2] For instance, in [**25**] it is crucial that every possible set of preferences be in the domain of their functions; in [**4**], a major assumption is that voters' preferences of subsets of candidates obey various (anti-)symmetric partial orders.

[3] Party primary systems are not neutral; over the *whole* election cycle, a candidate in an uncontested primary has (at least in principle) an advantage in winning the whole thing.

analysis since Condorcet, and one may note that a ranking of candidates is not simply a permutation, but an ordering.

The permutahedron has the right amount of structure for analyzing social preference functions while keeping pairwise behavior in mind. Its symmetry group sheds light on the structure of neutral SRSFs. The 'extra' symmetry of the permutahedron corresponds precisely to the well-known concept of *reversal* symmetry (see section 2.3). So the third goal of this paper is to prove significant results about neutral SRSFs utilizing the basic representation theory of the symmetry group of the permutahedron (summarized in the Appendix).

The most important result in this classification is an explicit characterization of the most symmetric possible rules in this family – a characterization which connects the two most important members of it.

Main Theorem. *If a neutral SRSF is compatible with pairwise information* and *fully preserves this information, then it is a rule along the one-parameter family of procedures connecting the Borda Count and the Kemeny Rule.*

'Pairwise information' means information about head-to-head comparisons between alternatives; see [**7**], Definition 4.6, and Theorem 5.12 for full details. By adding one final symmetry, one can characterize the Borda Count among SRSFs in the same way as is usually done among positional scoring rules, or rules relying only on pairwise information. Conversely, one can start moving beyond the 'Borda versus Condorcet' ways of thinking and start to explore *how much* of each behavior one might want in a choice procedure.

The remainder of the paper addresses the goals as follows:

- Review social choice definitions and introduce the permutahedron
- Motivate machinery with explicit statements and examples for $n = 3$
- Introduce all remaining needed concepts, and prove theorems for all n
- Look forward to questions opened up by this work, including other discrete structures of interest in social choice

2. Social Choice and Symmetry

2.1. General Definitions. We begin with relevant voting theory definitions, mostly using notation from the most relevant references ([**27, 5, 16, 17, 30**]).

Let $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ be a set of n candidates/alternatives; generic alternatives are given by capital letters such as A, B, C or X, Y, Z. Let $L(\mathcal{A})$ be the set of (strict) linear rankings of those alternatives, such as $A_1 \succ A_2 \succ \cdots \succ A_n$. Rankings correspond to permutations of the n elements of $\mathcal{A}$, and we will often identify $L(\mathcal{A})$ and S_n by abuse of notation; it should always be clear which is intended. Given a ranking r, if $X \succ Y$ in the order implied by r, we say that $X \succ_r Y$. Likewise, for any $1 \leq i \leq n$ the ith ranked alternative in r is denoted $r(i)$.

A *profile* $\mathbf{p}$ is a vector-valued function $\mathbf{p} : L(\mathcal{A}) \to \mathbb{Q}$, where one interprets each value as the number of voters[4] who prefer a given ranking in $L(\mathcal{A})$. Under this interpretation, the notation $\sum_{v \in L(\mathcal{A})} \mathbf{p}(v) f(v)$ signifies evaluating some function f over each ranking $v \in L(\mathcal{A})$ with multiplicity $\mathbf{p}(v)$, the 'number of voters preferring v in $\mathbf{p}$'. As an example, let $f(v) = 1$ if $v(1) = A$ and 0 otherwise; then this sum simply counts the number of voters putting A in first place.

[4] Although applications may have only integer numbers of voters, it is common practice to use $\mathbb{Q}$ to have a vector space, which also enables normalization, if desired.

A *social preference function* is a function from the set of all (finite) profiles to the power set of $L(\mathcal{A})$ (excepting the empty set). One might think of a social preference function as taking an electorate's set of preferences and yielding some nonempty set of rankings. The simplest social preference functions are obtained by taking a social welfare function or voting rule $\mathcal{F}$, and then returning the set of all full rankings which do not disagree with the outcome of $\mathcal{F}$.

DEFINITION 2.1. Let a *weighting vector* be an arbitrary[5] $\mathbf{w} \in \mathbb{R}^n$. Given a profile $\mathbf{p}$, we say an alternative X receives the *score* $\sum_{i=1}^{n} \sum_{v \in L(\mathcal{A}),\, v(i)=X} \mathbf{p}(v)\mathbf{w}(i)$ from the weighting vector; the set of alternatives with the maximum such score is the set of *winners*. The *positional scoring rule* associated to w is the social preference function which associates to each profile $\mathbf{p}$ all rankings r in which all winners are ranked above all non-winners.

Even viewed as preference functions, positional scoring rules are familiar. The vector $\mathbf{w} = (1, 0, 0, \cdots, 0)$ gives the plurality vote, and $\mathbf{w} = (n-1, n-2, \ldots, 1, 0)$ yields the *Borda Count* (BC).

2.2. Neutral Simple Ranking Scoring Functions. Our main objects of study are functions which essentially give scores to full rankings rather than individual candidates. The following definition is due to Conitzer et al. [**5**], though it is very similar to a roughly contemporaneous definition of generalized scoring rules in Zwicker [**35**] (in the case $I = O = L(\mathcal{A})$)[6].

DEFINITION 2.2. A social preference function f is a *simple ranking scoring function* (SRSF) if there exists a function $s : L(\mathcal{A}) \times L(\mathcal{A}) \to \mathbb{R}$ such that for all votes v, $f(v)$ is the ranking(s) r which maximizes $\sum_{v \in L(\mathcal{A})} \mathbf{p}(v)s(v, r)$. The function f is *neutral* if s is neutral; that is, if for any $\sigma \in S_{\mathcal{A}} = S_n$, $s(v, r) = s(\sigma(v), \sigma(r))$.

If an SRSF is neutral, then $s(v, r) = s(r, v)$, because we could have $\sigma = \pi^*$, where $\pi^*(v) = r$ and vice versa. For example, one could have s defined so that $s(v, r) = 1$ if $r = v$ and $s(v, r) = 0$ otherwise; in this case, we have the s.p.f. analogous to plurality, where the most popular ranking (or rankings) in the profile is the winning one.

The neutral SRSF concept is powerful, as it generalizes two otherwise disparate systems. One of these is the following often-studied (though less often used in practice) rule, as in Proposition 2 of [**5**].

DEFINITION 2.3. Let v and r be rankings, and $A, B \in \mathcal{A}$; then the following function measures agreement between v and r on the candidates A and B:

$$\delta(v, r, A, B) = \begin{cases} 1, & A \succ_r B \text{ and } A \succ_v B \\ 0, & \text{otherwise} \end{cases}.$$

The *Kemeny Rule* (KR) is the neutral SRSF with $s(v, r) = \sum_{A,B \in \mathcal{A}} \delta(v, r, a, b)$.

This definition is notationally dense (see [**14**] for the original, in terms of metrics). One should interpret this as saying that the Kemeny Rule evaluates a vote for the ranking v by assigning $\binom{n}{2}$ points to the ranking v, $\binom{n}{2} - 1$ points to any

[5]Usually one requires the entries to be nonincreasing as a function of index, but a priori this need not be so.

[6]Conitzer introduces these to study 'maximum likelihood estimators', while Zwicker puts them in a more general (and geometric) context.

ranking r differing by one switch of places from v (i.e. switching $v(i)$ and $v(i+1)$ for some i), and so on, down to no points to the ranking which reverses v completely. Then one adds up points as usual to determine the 'winning' ranking(s). KR is considered to be particularly important because it is the unique preference function which is a neutral and consistent Condorcet extension (see [**30**]).

EXAMPLE 2.4. Let's use the KR for a profile $\mathbf{p}$ for $n = 3$ with $\mathbf{p}(ABC) = 4$ (four voters choose ABC), $\mathbf{p}(BCA) = 3$, and no one chooses any other rankings ($\mathbf{p}(r) = 0$ for other rankings r). The maximum number of votes for a ranking is $\binom{n}{2} = 3$. In Figure 1 we can see how far distant each ranking is from the two with actual votes (for example, ACB is adjacent to ABC and CAB, even though the latter would seem far away in a single-winner context).

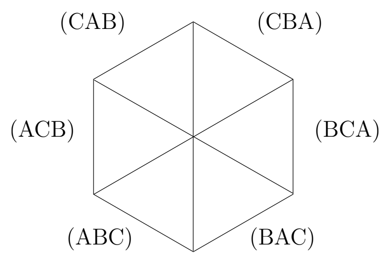

FIGURE 1

Put it together, and the winning ranking under KR is ABC:

ABC	$4 \cdot 3 + 3 \cdot 1$	$= 15$	CBA	$4 \cdot 0 + 3 \cdot 2$	$= 6$
BAC	$4 \cdot 2 + 3 \cdot 2$	$= 14$	CAB	$4 \cdot 1 + 3 \cdot 1$	$= 7$
BCA	$4 \cdot 1 + 3 \cdot 3$	$= 13$	ACB	$4 \cdot 2 + 3 \cdot 0$	$= 8$

On the other hand, every positional scoring rule is a neutral SRSF as well (Proposition 1 of [**5**]). Given a vote v and a candidate A, let $t(v, A)$ be the number of points that A gets if someone votes v. Then if we denote the ith-place candidate in a ranking r by $r(i)$, the function

$$s(v, r) = \sum_{i=1}^{n} (n - i) t(v, r(i))$$

turns a positional scoring rule into an SRSF. Intuitively, the SRSF score for v with respect to a ranking r is the sum of points each candidate in ranking r gets for vote v in the scoring rule, weighted by the position of the candidate in the ranking r.

We can make all this quite concrete with three candidates. For a positional scoring rule, if $v = XYZ$ and $r = ABC$, then

$$\begin{aligned} s(v, r) &= \sum_{i=1}^{3} (3 - i) t(XYZ, r(i)) \\ &= 2 \cdot t(XYZ, A) + 1 \cdot t(XYZ, B) + 0 \cdot t(XYZ, C) \\ &= 2t(XYZ, A) + t(XYZ, B)\,, \end{aligned}$$

so if we have a system with $\mathbf{w} = (u, w, 0)$, then this yields

v	ABC	ACB	CAB	CBA	BCA	BAC
$s(v, ABC)$	$2u + w$	$2u$	$2w$	w	u	$2w + u$

Figure 2 gives a visualization using a Saari-like triangle, where each separating line defines the border between rankings with $X \succ Y$ and vice versa. To use it for other $v' \neq v$, one would permute the whole triangle with the permutation σ such that $\sigma(v) = v'$. Then computing the SRSF for each $r = XYZ$ can be done visually as well, by taking the dot product of the profile and the weighting triangle with ABC on the XYZ spot.

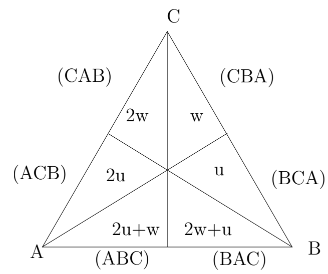

FIGURE 2. Visualizing positional scoring rules as SRSFs

EXAMPLE 2.5. It is instructive to see what plurality looks like as an s.p.f. Since $t(v, r(i)) = 0$ unless $r(i) = v(1)$, in which case we get $s(v, r) = n - i$, the score for r is $\sum_{v \in L(\mathcal{A})} \mathbf{p}(v)s(v, r)$, which is the sum of $n-1$ points for each voter who ranks $r(1)$ first, $n-2$ points for each one who ranks $r(2)$ first, and so forth.

For $n = 3$, with the profile from Example 2.4, we see that ABC is again the aggregate preference.

r	$\sum_{v \in L(\mathcal{A})} \mathbf{p}(v)s(v, r)$	
ACB	$4 \cdot 2 + 3 \cdot 0$	$= 8$
ABC	$4 \cdot 2 + 3 \cdot 1$	$= 11$
BAC	$4 \cdot 1 + 3 \cdot 2$	$= 10$
BCA	$4 \cdot 1 + 3 \cdot 0$	$= 4$

EXAMPLE 2.6. On the other hand, the Borda Count gives BAC as the winning ranking. Putting $u = 2$ and $w = 1$ gives the following SRSF-style scores.

r	$\sum_{v \in L(\mathcal{A})} \mathbf{p}(v)s(v, r)$	
ACB	$4 \cdot 4 + 3 \cdot 1$	$= 19$
ABC	$4 \cdot 5 + 3 \cdot 2$	$= 26$
BAC	$4 \cdot 4 + 3 \cdot 4$	$= 28$
BCA	$4 \cdot 2 + 3 \cdot 5$	$= 23$

Just as the analysis of [**7, 18, 19**] considers the point totals to be vital information in understanding voting function symmetry, we consider the point totals for SRSFs to be vital to unlocking the structure of social preference functions.

2.3. The Permutahedron and Reversal Symmetry. The input profiles and output scores of SRSFs are both essentially elements of an $n!$-dimensional vector space over $\mathbb{Q}$. Since we identify rankings with permutations, we identify this space with the group ring $\mathbb{Q}S_n$, which is the set of all formal $\mathbb{Q}$-sums $\sum_{\sigma \in S_n} q_\sigma \sigma$. To be specific, we are equating σ with the ranking r such that $r(\sigma(i)) = X_i$.[7] In

[7]For instance, $\sigma = (1\,2\,3)$ will correspond to $r(1) = r(\sigma(3)) = X_3$, $r(2) = r(\sigma(1)) = X_1$, and $r(3) = r(\sigma(2)) = X_2$, or $X_3 \succ X_1 \succ X_2$.

addition, *neutral* SRSFs have *all* their $s(v,r)$ given by *one* vector in this same space, since $s(\sigma(v),r) = s(v,\sigma^{-1}(r))$. Hence, to analyze neutral SRSFs, we will want to look at this structure. See Section 3 for concrete examples when $n = 3$.

It is time to introduce the other major player in our story. Traditional analyses of both positional scoring rules and the Kemeny rule involve yet another symmetry – the concept that if everyone reverses *all* their preferences, then the final outcome should be reversed as well.

Let $\rho = (1,n)(2,n-1)\cdots$ be the so-called 'reversal' element of S_n. Then the ranking corresponding to $\rho\sigma$ is the one such that $r(\rho\sigma(i)) = r(n+1-\sigma(i)) = X_i$, or in other words the strict reversal of the ranking corresponding to σ (like $A \succ B \succ C$ is the reversal of $C \succ B \succ A$). For a general ranking v, we denote its reversal by v^ρ; we will use the same notation for the operation of reversing all rankings in a set or changing $\mathbf{p}(v)$ to $\mathbf{p}(v^\rho)$ for all v in a profile.

Definition 2.7. We say a social preference function f has *reversal symmetry* if $[f(\mathbf{p})]^\rho = f(\mathbf{p}^\rho)$ for all profiles $\mathbf{p}$.

Not all SRSFs observe this symmetry, not even simple ones like plurality. Any profile $\mathbf{p}$ with 25% each preferring $A \succ B \succ C$, $A \succ C \succ B$, $C \succ B \succ A$, and $B \succ C \succ A$ suffices, as with plurality the winning rankings are ABC and ACB whether one uses $\mathbf{p}$ or $\mathbf{p}^\rho$. Examples of rules which *do* have reversal symmetry are BC and KR. They are symmetric with respect to the following combinatorial object.

Definition 2.8. The *n-permutahedron* Π_n is the graph with $n!$ vertices, indexed by permutations of the set $\{1,2,\ldots,n\}$ (or elements of S_n, as preferred), and with an edge connecting permutations σ and σ' if and only if $\sigma' = (i,i+1)\sigma$ for some $1 \le i < n$. (See [**32**] for more details.) We call its symmetry group P_n.

Another way to say this same definition is that the permutahedron is the Cayley graph of the symmetric group S_n for the neighbor-swap generating set

$$\{(1,2),(2,3),\ldots,(n-1,n)\}\ .$$

Since ρ simply changes i to $n+1-i$ in a permutation, a neighbor-swap $(i,i+1)$ will become $(n+1-i,n+i)$, so all edges are preserved by ρ, which means this graph is just the tool for looking at reversal symmetry.

The 3-permutahedron is in fact the (graph associated to the) regular hexagon, where the vertices are labeled by permutations of $\{1,2,3\}$, written as reduced words. Just in the case of dimension three this also works by labeling edges instead, so we do this in Figure 3 because of the useful analogies with the representation triangle and Figure 1. This helps visualize neutral SRSFs which satisfy reversal symmetry. Think of the score for a ranking r as being a sort of 'dot product' of the profile with the hexagon, except that rankings the same distance from r get the same score.

For example, Figure 4 gives the vector of weights and the profile for Example 2.4. Imagine rotating the XYZ hexagon so that XYZ is on each ranking on the right; the sum of the products of each region will be the same as before.

Figure 5 shows the 4-permutahedron (where vertices are again labeled).

Reversal symmetry is important, but to ensure no symmetries are missed, one needs the overall symmetry group – that is, what is P_n? The answer is more mathematical folklore[8] than otherwise, but it turns out that $P_n \cong S_n \times C_2$, where

[8] See [**31, 9, 6**] for proofs and discussion.

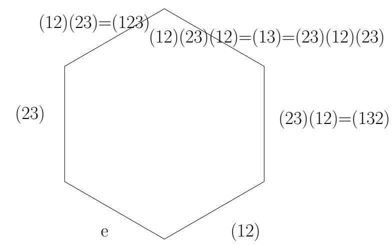

FIGURE 3. The 3-permutahedron

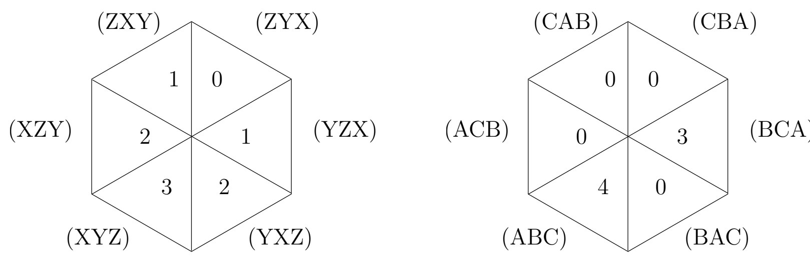

FIGURE 4. Visualizing the scoring of the Kemeny Rule

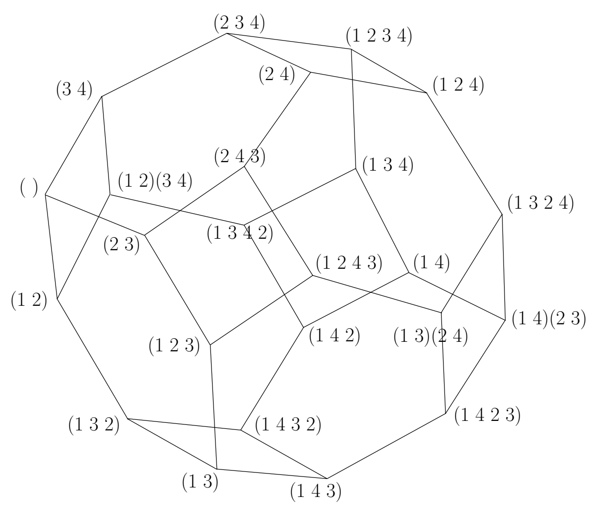

FIGURE 5. The 4-permutahedron

C_n is the cyclic group of order n. The C_2 subgroup is precisely[9] that given by ρ, the reversal symmetry!

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[9]Since it has index two, it is clear that $S_n \triangleleft P_n$. In particular, if one thinks of S_n as acting by right multiplication on the permutahedron, this provides a natural inclusion inside P_n. Then, since we already saw that ρ acts on the *left* on the permutahedron, the action given by reversal symmetry defines exactly the subgroup C_2 which gives a direct product, since left and right multiplication will commute.

3. Decompositions and Voting

The full power of representation theory for analyzing neutral SRSFs requires enough machinery that we postpone it to Section 4. In this section, we use this power *implicitly*, and motivate our theorems with detailed examples, preliminary assertions, and informal proofs when $n = 3$.

Recall that the profile space in question is isomorphic to $\mathbb{Q}S_3$, a six-dimensional space. Importantly, the image of an SRSF *also* will be a subspace of $\mathbb{Q}S_3$. That means the decomposition of the profile space into Basic/Borda, Reversal, Condorcet, and Kernel components, first fully introduced in [**17**], can be given an explicit basis in $\mathbb{Q}S_3$.

K	$(1,1,1,1,1,1)$	$S^{(3)}$
B_A	$(2,2,0,-2,-2,0)$	$S_1^{(2,1)}$
B_B	$(0,-2,-2,0,2,2)$	
R_A	$(1,1,-2,1,1,-2)$	$S_2^{(2,1)}$
R_B	$(-2,1,1,-2,1,1)$	
C	$(1,-1,1,-1,1,-1)$	$S^{(1,1,1)}$

We use group elements in the order $e, (2\,3), (1\,2\,3), (1\,3), (1\,3\,2), (1\,2)$[10]. The representation-theoretic notation for the subspaces is in the right column; this notation makes it clear the Reversal and Borda components, whose basis elements are not orthogonal to each other, must be considered as inherently two-dimensional. (The use of B_X and C to indicate profiles should not cause ambiguity with generic candidate names.) Finally, note that the sum of the entries of each vector (except the first) is zero; such a vector is called *sum-zero*, and such profiles are called *profile differentials*, inasmuch as they do not represent actual voters.

Let f be a neutral SRSF. Recall (Subsection 2.2) that f is uniquely defined by all its $s(\cdot, r)$, which we may consider to be a vector of weights $\mathbf{s}$; we will call the function $f_{\mathbf{s}}$ to indicate this fact. Thus the scores for all rankings r are simply the dot products $\sigma(\mathbf{s}) \cdot \mathbf{p}$ from before, so $f_{\mathbf{s}}$ gives a linear transformation from $\mathbb{Q}S_3$ to itself. This is given by the scores for each ranking as in the previous section[11].

But $f_{\mathbf{s}}$ is not just any linear transformation. Since we pointed out earlier that $s(v, r) = s(\sigma(v), \sigma(r))$ for any permutation $\sigma \in S_n$, $f_{\mathbf{s}}$ must preserve all group symmetries; so, $f_{\mathbf{s}}$ is what is called an S_3-module homomorphism (see Section 4.1 for more detail). This means SRSFs are subject to the following basic fact:

Schur's Lemma for $n = 3$. *The image of any of K, B, R, or C under a neutral SRSF will only be a (possibly zero) multiple of itself, except B_X and R_X may be sent to linear combinations of each other*[12].

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[10] Corresponding to the usual voting theory order ABC, ACB, CAB, CBA, BCA, BAC.

[11] So that for KR we would have the matrix $\begin{pmatrix} 3 & 2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 3 & 2 & 1 \\ 1 & 2 & 3 & 2 & 1 & 0 \\ 2 & 3 & 2 & 1 & 0 & 1 \end{pmatrix}$.

[12] The X basis vector of these components must go to a linear combination of the X vectors (or their orthogonal complements within the B and R modules). This is because these vectors have symmetry under any σ which switches alternatives Y and Z, while the others do not, and $f_{\mathbf{s}}$ is an S_3-module homomorphism.

The projections of any profile onto these subspaces also are constrained by Schur's Lemma. So if we count dimensions, that means a general neutral SRSF has six degrees of freedom, which makes sense since $\mathbf{s}$ has six independent entries.

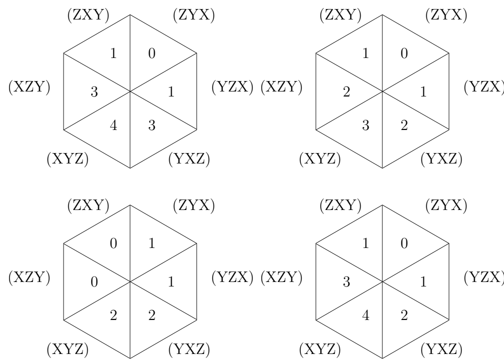

FIGURE 6. Various SRSFs – Borda, Kemeny, 'Nonsense', Borda Variant

EXAMPLE 3.1. Figure 6 shows a few examples of weighting vectors. The first two are the BC (rescaled) and the KR, which have already been met before.

The third appears to be an amusing nonsense procedure where $\mathbf{s} = (2, 0, 0, 1, 1, 2)$. Here, $f_{\mathbf{s}}(B_A) = -2R_A$; that is, a voter profile which overwhelmingly approves of rankings with A first ahead of other rankings would have a result overwhelmingly favoring any ranking with A in second place!

On the other hand, the last procedure (with $\mathbf{s} = (4, 3, 1, 0, 1, 2)$) is a 'reasonable' variant on the Borda Count which is trying to imitate plurality a little bit by deemphasizing YXZ as an outcome by voters who chose XYZ – perhaps with a view toward making sure ACB is the outcome more often with profiles like the one from Example 2.4. (It is *not* a positional scoring rule.)

Nonetheless, this $\mathbf{s}$ has a nonzero dot product with the $\mathbf{s}$ in the third procedure, so for some profiles with a large B_A component relative to the others, it will exhibit much the same bizarre behavior and should also be called into question.

Our main interest is in decomposing the *images* of SRSFs, but the algebra also identifies building blocks for sensible procedures. The following table gives basis vectors for $\mathbf{s}$ which send each component (subspace) only to a scalar multiple of itself and kill everything else.

K	$(1, 1, 1, 1, 1, 1)$
C	$(1, -1, 1, -1, 1, -1)$
B	$(2, 1, -1, -2, -1, 1)$
R	$(2, -1, -1, 2, -1, -1)$

The other two dimensions' worth of $\mathbf{s}$ are not reasonable – for instance, the third procedure in Example 3.1 is a basis for *any* procedure which behaves like it.

Notice that simply rescaling the Borda Count weighting vector so it is sum-zero gives the prototype for methods which preserve only the Basic component, as expected from [**18**]. In fact, by ignoring the part of $f_{\mathbf{s}}$ coming from K above in a systematic way (because it will always add the same amount to each ranking r), we can reduce our attention to sum-zero $\mathbf{s}$. Likewise, we only care about *relative* scores, which leads to the following definition (as in [**7**], [**17**], and elsewhere).

DEFINITION 3.2. We consider two procedures $f_{\mathbf{s}}$ and $f_{\mathbf{s}'}$ with sum-zero weighting vectors $\mathbf{s}, \mathbf{s}'$ to be *equivalent* if $\mathbf{s} = k\mathbf{s}'$, which we will indicate by $s \sim s'$. We will say that two neutral SRSFs are *essentially different* if they are *not* equivalent in this way.

PROPOSITION 3.3. *The space of essentially different neutral SRSFs for $n = 3$ is four dimensional.*

INFORMAL PROOF. There are six total dimensions. Taking the quotient by K to get a sum-zero weighting vector removes one; unscaling removes another. □

In this case, plurality has $\mathbf{s} \sim (1, 1, -1, -1, 0, 0)$. We will consider the 'reversed' plurality $\mathbf{s}' \sim (-1, -1, 1, 1, 0, 0)$ to be equivalent, since it will literally have a reversed outcome which can be derived from plurality – even though it looks quite different to the voter. (We are intentionally ignoring issues such as unanimity to get the broadest possible result at this point.)

Many theorists argue that the Condorcet (C profile) component should be ignored (or, what is equivalent, considered a complete tie). The idea is that any profile non-orthogonal to this component runs the risk of giving credence to subspaces where each candidate appears in each position in the ranking an equal number of times. If we do ignore it, we lose another dimension:

PROPOSITION 3.4. *The space of neutral SRSFs for $n = 3$ which ignore the Condorcet component is three dimensional.*

In the contexts mentioned in the introduction, insisting on this restriction does not always make sense. The famous Condorcet example of $\mathbf{p}$ with $\mathbf{p}(ABC) = \mathbf{p}(BCA) = \mathbf{p}(CAB) = 1$, $\mathbf{p}(XYZ) = 0$ otherwise need not be a paradox in the committee example; there, it is plausible that the voters would prefer to have a random tiebreaker among just these three rankings (as opposed to all six), guaranteeing at least one of their succession preferences.

Naturally, not every application will demand preserving the Condorcet component, and we are not arguing that the Condorcet criterion or Condorcet extensions are always appropriate. Rather, it seems reasonable that in situations where the overall ranking matters more than the winner, or where there is potential for the ranking to influence (or even determine) a cycle of events, it is advantageous to keep this component[13].

One of the ways in which we can ensure that we do not throw away this information is by means of the concept of being *compatible with pairwise information.* Definition 4.6 gives a full account, but for $n = 3$ it is sufficient to remark that such an $f_{\mathbf{s}}$ kills everything from R and K; the intuition is that only B and C preserve the information we get from tallying all the head-to-head pairwise matchups between candidates. By counting dimensions once again one can compute that, modulo equivalence:

PROPOSITION 3.5. *The space of neutral SRSFs for $n = 3$ which are compatible with pairwise information is two dimensional.*

Unfortunately, the procedure with $\mathbf{s} = (2, 0, 0, 1, 1, 2)$ (recall, where $f_{\mathbf{s}}(B_A) = -2R_A$) is in this space. So this is not a panacea.

[13]We briefly mention *cyclic orders* in Example 6.1.

Now we bring in the permutahedron, with its reversal symmetry. All of the components of the decomposition of $\mathbb{Q}S_3$ have a natural action of ρ as well (so they are "P_3-modules"). It is not hard to see that $(B_X)^\rho = -B_X$ while $(R_X)^\rho = R_X$, so although B and R are equivalent under S_3, they are *not* equivalent under reversal. The implication in the social preference function context is that with reversal symmetry, B_X and R_X must not go to each other. Thus we have:

Proposition 3.6. *The space of essentially different neutral SRSFs for $n = 3$ which obey reversal symmetry is two dimensional.*

Informal Proof. There are six total dimensions, and as usual we eliminate two by considering sum-zero essentially different procedures. Ordinarily, a basis element B_X of the basic subspace could be sent to some element of the reversal space, and vice versa, but if not, then quotienting out eliminates two additional dimensions. □

The bizarre SRSF $\mathbf{s} = (2, 0, 0, 1, 1, 2)$ is not allowed, nor is anything which shares a nontrivial piece of it. However, SRSFs having reversal symmetry lead to the same problems one gets from positional scoring procedures which do not ignore the reversal component. Here is a somewhat subtle example – again, this is *not* a positional scoring procedure.

Example 3.7. The weighting vector $\mathbf{s} = (15, 2, 0, 11, 0, 2)$ puts appropriately heavy weight on XYZ and some weight on its neighbors. Consider the profile $\mathbf{p} = (9, 6, 6, 3, 0, 6)$ with these weights; ABC (the first ranking in the profile) will be given $15 \cdot 9 + 2 \cdot 6 + 0 \cdot 6 + 11 \cdot 3 + 0 \cdot 0 + 2 \cdot 6 = 192$ points, while BAC (the last ranking in the profile) receives $15 \cdot 6 + 2 \cdot 0 + 0 \cdot 3 + 11 \cdot 6 + 0 \cdot 6 + 2 \cdot 9 = 174$.

The final score vector is $(192, 120, 174, 156, 84, 174)$; note that ACB loses by a significant margin, even to CBA, despite the pairwise tally showing A the clear victor and B tied with C.

With counterintuitive results coming no matter what restrictions we place on the symmetry, what happens if we demand maximum symmetry from a neutral SRSF?

Proposition 3.8. *The space of essentially different neutral SRSFs for $n = 3$ which obey reversal symmetry* and *are compatible with pairs is a one-dimensional family of procedures.*

For $n = 3$, one may think of this as giving the space of $f_{\mathbf{s}}$ of the sum-zero vectors of weights in Figure 7.

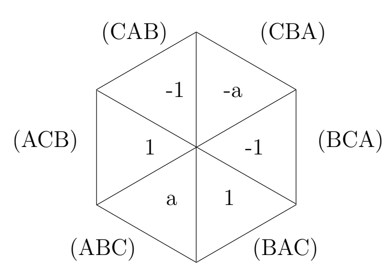

Figure 7. The continuum of weights for Borda-Kemeny line, $n = 3$

Clearly both BC ($a = 2$) and KR ($a = 3$) are part of this continuum; since it is one-dimensional, they define it as well. This is our main result in the case $n = 3$.

Using a vector of weights from Figure 7 alone can still lead to a nonsense method; for example, letting $a = 0$ gives something less than useful. However, it is very hard to decide what sort of other, *non-algebraic*, conditions are natural. One can impose unanimity-style conditions such as $a \geq 1$, under which any profile in which all voters have the same preference yields that preference as a winning ranking. Nonetheless, even then it is possible for SRSFs in this space to give actual outcomes (winning preference orders) which are different from both BC and KR.

Example 3.9. The profile $\mathbf{p} = (1, 2, 5, 0, 0, 0)$ only has non-zero preference for half the rankings (ABC, ACB, and CAB). For a given a, this means the scores for the relevant potential winning rankings will be:

	Computation	BC	KR	$a = 1.5$
ABC	$+a \cdot 1 + 1 \cdot 2 - 1 \cdot 5$	0	-1	-1.5
ACB	$+1 \cdot 1 + a \cdot 2 + 1 \cdot 5$	12	10	9
CAB	$-1 \cdot 1 + 1 \cdot 2 + a \cdot 5$	16	11	8.5
CBA	$-a \cdot 1 - 1 \cdot 2 + 1 \cdot 5$	0	1	2

The Kemeny Rule and Borda Count both give CAB as the winning outcome, but with $a = 1.5$ the result is ACB.

However, it turns out that if $2 < a < 3$, this is not possible – all SRSFs of this type will have the same outcome as KR or BC (or both). Demonstrating this is a standard and tedious chase of inequalities to yield contradictions from other cases, so we omit the proof.

For the reader who enjoys an exercise, here is an unscaled, non-zero-sum (and hence more intuitive to the layman) example of an SRSF 'between' the most familiar examples.

Example 3.10. The procedure in Figure 8 yields a tie between ABC and ACB (the BC and KR outcomes, respectively) on the profile $\mathbf{p} = 6B_A + 2B_B - 7C - 3R_C + 12K$.

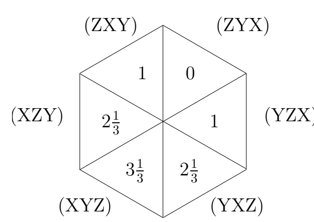

Figure 8. A procedure between Borda and Kemeny

Hints: which $2 < a < 3$ would this correspond to? What contribution will K make in the finally tally? What about C? Can you now reconstruct the *relevant* part of the profile and finish the computations as in Example 3.9?

These examples show that there is real depth in the concept of simple ranking scoring functions. In order to avoid the problem in Example 3.9, we must use algebra – for instance, we could send the profile differential C to a *positive* scalar multiple of itself. To ensure that the outcome is 'between' those of the well-known methods (if this is even a good idea), we must come up with appropriate Pareto-type conditions.

It is by no means obvious how much weight to give the C component, but once one bothers with the pairwise information, its effects must be considered. If there is some dissatisfaction in the community about the Borda Count completely ignoring this information, there is also dissatisfaction with methods that give it as much weight as the Kemeny Rule does, frequently exhibiting paradoxes as the no-show paradox.

With this spectrum, each potential 'customer' of methods on this 'Borda-Kemeny spectrum' can decide this for themselves how much or how little to take this into account; the algebraic access to these procedures makes this analysis possible.

4. Representations

Let $\mathcal{A}$ be a set of n candidates, S_n be the symmetric group on n elements, and so forth. Our results for $n \geq 3$ may be summarized as follows.

- The space of essentially different neutral SRSFs which are compatible with pairs is $\frac{1}{2}(n+1)(n-2) = \frac{1}{2}(n^2 - n + 2)$ dimensional (Theorem 5.1).
- If these also have reversal symmetry, we are reduced to $\frac{1}{4}(n^2 - 5)$ or $\frac{1}{4}(n^2 - 4)$ dimensions for odd and even n, respectively, which is about half as many (Theorem 5.4).
- Unsurprisingly, there are n dimensions of positional scoring rules, $\left\lfloor \frac{n}{2} \right\rfloor$ of which obey reversal symmetry (Theorems 5.5 and 5.6).
- An SRSF which is a scoring rule *and* is pairwise compatible is essentially the same as the Borda Count or its reversal (Corollary 5.9).
- Consider SRSFs which are pairwise compatible *and* preserve the information from pairwise matchups as fully as possible. There is just one dimension of essentially different neutral SRSFs in this family, and it is defined by the continuum connecting the Borda Count and the Kemeny Rule (Theorem 5.12).

The main importance of most of these theorems is to emphasize *how many* different SRSFs there are if we only use *some* symmetry. The number grows as $O(n^2)$ for most – the curious reader may also skip ahead to Theorem 7.1 for full, precise details and tables of the dimensions of the decompositions. Given that the number of pairwise votes that can happen grows at this rate as well, this is not surprising; nonetheless, it serves to highlight the surprising parsimony of Theorem 5.12 and Corollary 5.9.

In order to explain these theorems in full generality in Section 5, we will first give a brief review of how representation theory, modules, and voting theory can interact, then give the relevant parts of the representation theory of the permutahedron for all n.

4.1. Representation Theory, Modules, and Voting. For SRSFs, recall that we may consider both the domain and target of a given function $f_{\mathbf{s}}$ to be a vector space of dimension $n!$. Because of the role of permutations in the voting context, we will in fact consider the vector space to be the group ring $\mathbb{Q}S_n$; this has a natural S_n action by concatenating permutations. To keep notation consistent with [**7**], we will also often call this vector space $M^{1,1,\cdots,1} = M^{1^n}$; for any partition λ of n, there is a corresponding module M^λ of profiles of preferences on n candidates with ties, but here we will only consider ones with no ties, coming from the maximal partition $\lambda = (1, 1, \cdots, 1)$.

As before, for any neutral SRSF f, we can let $\mathbf{s}$ be the vector of all $s(\cdot, r)$ and specify $f = f_{\mathbf{s}}$; then $f_{\mathbf{s}}$ is really a linear transformation $f_{\mathbf{s}} : M^{1^n} \to M^{1^n}$. Is there any *group* structure here? The answer is yes.

- By definition, SRSFs only depend on the number of each ranking in the profile $\mathbf{p}$ (they are *anonymous*). Hence if we define $\sigma(\mathbf{p}(v)) = \mathbf{p}\left(\sigma^{-1}(v)\right)$, the outcome of the SRSF will change according to σ as well. That is, the profile space M^{1^n} is a $\mathbb{Q}S_n$-module. The same is true for the outcome space.
- But by having *neutral* SRSFs with $s(v, r) = s(\sigma(v), \sigma(r))$, we see that the action of σ propagates from profiles to outcomes. That is, for each r,
$$\sigma\left(\sum_{v \in L(\mathcal{A})} \mathbf{p}(v) s(v, r)\right) = \sum_{v \in L(\mathcal{A})} \mathbf{p}(\sigma^{-1}(v)) s(\sigma^{-1}(v), \sigma^{-1}(r)) ,$$
which is the same as the effect of σ on the final ranking, so $f_{\mathbf{s}}$ is a $\mathbb{Q}S_n$-module homomorphism, essentially by definition. Every single voting procedure under discussion *is* a group-theoretic object.
- We have even more; by exactly the same argument as in [**7**], since $\mathbf{p} \in M^{1^n} \cong \mathbb{Q}S_n$, a neutral SRSF is the result of the profile acting on $\mathbf{s}$, so that $f_{\mathbf{s}}(\mathbf{p}) = \mathbf{ps}$, in the sense of the group rings.

Once we know that $f_{\mathbf{s}}$ is a $\mathbb{Q}S_n$-module homomorphism, we can use representation theory to find out things about it. Our main tool will be decomposition into irreducible submodules, and the following well-known result:

Schur's Lemma. *Let G be a group. If M and N are irreducible G-modules and $g : M \to N$ is a G-module homomorphism, then either $g = 0$ or g is an isomorphism.*

Thus far, $G = S_n$ for us. It is a classical result in representation theory of the symmetric group that the irreducible modules of S_n are indexed by the partitions of n; see the Appendix for more details and their dimensions. More importantly, the irreducible decomposition of $\mathbb{Q}S_n = M^{1^n}$ is given by the sum of a number of each of these irreducible modules, k for a k-dimensional one. For $n = 3$, this was
$$M^{(1,1,1)} \cong S^{(3)} \oplus 2S^{(2,1)} \oplus S^{(1,1,1)}$$
and corresponded directly with the K, B, R, and C components.

Let's use this to further justify some of the claims in Section 3. Since any $f_{\mathbf{s}}$ is a S_3-module homomorphism, Schur's Lemma tells us that $S^{(3)}$ (K) and $S^{(1,1,1)}$ (C) go to themselves in any neutral SRSF. In both cases this will be by multiplication by some scalar. On the other hand, each $S^{(2,1)}$ can go to any linear combination of the two $S^{(2,1)}$ components – that is, B_A could go to any combination $xB_A + yR_A$.

However, these spaces are also P_n-modules, provided we restrict to procedures with reversal symmetry. Schur's Lemma now distinguishes between the two copies of $S^{(2,1)}$ given by B and R, because $(B_X)^\rho = -B_X$ while $(R_X)^\rho = R_X$.

4.2. Voting-Theoretic Decompositions for General n. We now move to the decomposition of the profile space $M^{1^n} \cong \mathbb{Q}S_n$ as S_n- and P_n-modules. Referring to the Appendix, we note that the canonical irreducible decomposition of M^{1^n} has $n-1$ copies of $S^{(n-1,1)}$ and $\binom{n-1}{2}$ copies of $S^{(n-2,1,1)}$. These are the only ones we will need to deal with.

There are various ways to think of $(n-1)S^{(n-1,1)}$, but the classification in [**18, 19**] as the Borda (B), Alternating (Alt), and Symmetric (Sym) components is most useful. As with our $n=3$ example, these are profile *differentials* (i.e. they are sum-zero), since they are orthogonal to $S^{(n)}$.

We summarize them in the following table, with unmentioned values being zero. Each irreducible component is the direct sum of one-dimensional vector spaces given by B_X, $Alt_{j,X}$, or $Sym_{j,X}$ (for any given j) for all candidates X.

B_X	$B_X(r) = n+1-2k$	if $r(X) = k$
$Alt_{j,X}$	$Alt_{j,X}(r) = n-1$	if $r(X) = j$
$2 \leq j \leq \frac{n}{2}$	$Alt_{j,X}(r) = 1-n$	if $r(X) = n+1-j$
	$Alt_{j,X}(r) = 2j-n-1$	if $r(X) = 1$
	$Alt_{j,X}(r) = 1+n-2j$	if $r(X) = n$
$Sym_{j,X}$	$Sym_{j,X}(r) = 1$	if $r(X) = j$ or $n+1-j$
$2 \leq j < \frac{n+1}{2}$	$Sym_{j,X}(r) = -1$	if $r(X) = 1$ or n
$Sym_{\frac{n+1}{2},X}$	$Sym_{\frac{n+1}{2},X}(r) = 2$	if $r(X) = \frac{n+1}{2}$
	$Sym_{\frac{n+1}{2},X}(r) = -1$	if $r(X) = 1$ or n

For example, when $n=4$, the Borda component for A has $3, 1, -1, -3$ voters for rankings with A in first through fourth place respectively. That is, B_A is a profile (differential) with 3 voters each for $ABCD$, $ABDC$, $ACBD$, $ACDB$, $ADBC$, and $ADCB$, but -1 votes each for $BCAD$, $BDAC$, $CBAD$, etc., and so forth. The profile $Alt_{2,A}$ has $-1, 3, -3, 1$ for the same places[14]; the analogous symmetric profile grants $-1, 1, 1, -1$ votes, respectively, to rankings with A in first through fourth places.

In all cases, these have the structure that the sum over *all* candidates of each of these profiles is zero, so that each component, as a vector space, is $(n-1)$-dimensional. We can start to see what role these play with the following examples.

EXAMPLE 4.1. Let's see what happens to the Borda and Sym components under plurality for $n=4$. Recall that if $r = XYZW$, plurality is the SRSF with $\mathbf{s} = s(\cdot, r)$ giving 3 weighting points for any ranking with X in first place, 2 for Y in first, 1 for Z and 0 for W.

What happens to B_A under this system? Any ranking r of the form $AYZW$ will have 18 total compatible voters in B_A (three each of the six possible permutations with A in first place) giving 3 points each. How many voters will give 2, 1, or 0 points to r? Once we pick Y, Z, or W to be in first place, there are two of each kind of those voters with A in second, third, and fourth place, respectively, weighted by $1, -1, -3$ in the profile – giving $2(1-1-3)$ 'total voters' giving each 2, 1, and 0 points. Subtracting this from 54 yields 36 points for r of the form $AYZW$.

In the same manner, any r of the form $XAZW$ has the same 18 voters giving *two* points each, and the rest giving 3, 1, and 0 points each, for 12 points per this type of ranking r; by symmetry, we see that B_A will be sent to $12B_A$ by the plurality function.

EXAMPLE 4.2. Using the same strategy, the symmetric component S_A has -6 $(= -1 \cdot 6)$ voters granting 3 points each to $r = AYZW$. Considering again the two

[14] So -1 votes for $ABCD$, $ABDC$, $ACBD$, $ACDB$, $ADBC$, and $ADCB$, ...

rankings each with Y, Z, or W first and A in the other spots, we have $2(1+1-1)$ voters giving 2, 1, and 0 points, for a total of -12 points. One can compute that rankings of the form $r = XAZW$ will receive -4 points each, and by symmetry we see that S_A is sent to $-4B_A$ by plurality.

But the B, Alt, and Sym components also happen to be irreducible P_n-modules! It is not hard to see that under reversal symmetry the B and Alt components reverse sign, while the Sym components are unchanged, so we use the notation $S^{(n-1,1),+}$ for the isomorphism class of the Sym modules, while the B and Alt components are called $S^{(n-1,1),-}$.

What about generalizing the discussion for $n = 3$ to the components of $S^{(n-2,1,1)}$? Most of these vanish under even the weakest symmetry we'll discuss, so the final component to describe corresponds to $(1,-1,1,-1,1,-1)$ when $n = 3$.

Definition 4.3. For each pair $\{X, Y\}$ of candidates, we will define C_{XY}, the *Condorcet component*, as follows[15]:

Let $\{XY1\}$ denote the set of all rankings which begin $X \succ Y$, let $\{XY2\}$ denote the set of all rankings which begin with $X \succ ? \succ Y$, and continue up through $\{XY(n-1)\}$. Then C_{XY} is the profile where, for all cyclic permutations of the elements in $\{XYi\}$ (such as ABC, BCA, CAB for three candidates), we assign $n - 2i$ voters to those rankings.

Notice that $\{XYi\}$ is simply the reversal of all $\{XY(n-i)\}$, so there is redundancy. For $n = 3$, this does give the usual Condorcet component, while for $n = 4$ it gives the Condorcet component in the form of the C_{XY}s of Saari, as for $i = 2$ we get zero voters. A convenient basis of dimension $\binom{n-1}{2}$ is given by

$$C_{A_1A_2}, C_{A_1A_3}, \cdots, C_{A_1A_{n-1}}, C_{A_2A_3}, \cdots, C_{A_2A_{n-1}}, \cdots, C_{A_{n-2}A_{n-1}}$$

where one notes that holding X *or* Y constant and summing over all candidates in the other variable gives zero.

4.3. Pairwise Compatibility. In order to pursue the finest-grained results, we need one last set of concepts. The first three are directly from [**7**].

Definition 4.4. We define the *pairs map* $P : M^{1^n} \to M^{1^n}$ to be the linear transformation that sends a basis vector of M^{1^n} to the sum of all such vectors whose top two candidates are in the same order as in the input vector.

For instance, if $\mathbf{p}(BACD) = 1$ and $\mathbf{p}(XYZW) = 0$ for all other rankings, $P(\mathbf{p})(XYZW) = 1$ if XY is one of the (ordered) pairs BA, BC, BD, AC, AD, CD and $P(\mathbf{p})(XYZW) = 0$ otherwise. This naturally encodes all the usual information we associate with comparing candidates on a pairwise basis – for instance, in the Borda Count and Kemeny Rule.

Definition 4.5. The *effective space* of a linear transformation T is the orthogonal complement to the kernel of T. This determines what will *not* be in the kernel.

[15] This is obviously indebted to Saari's original Condorcet component and Zwicker's 'spin' component [**33**] as well as Saari's C_{XY} in Section 6 of [**18**]. Indeed, the spaces only differ when $n \geq 5$, which is probably why they are first completely described here. See also Sections 4.4.3 and 4.5 of [**20**] where they are implicit in a discussion about the 'old' Condorcet components.

In essence, this is the subspace of elements of the domain of T (and in our context, f) which have no part simply sent to zero (in our context, a complete tie). Since we can often compute the dimension of this space, it will help us compute the dimensions of the sets of procedures. In particular (see the discussion before Theorem 6 of [**7**]), the effective space of P is isomorphic as an S_n-module to $S^{(n)} \oplus S^{(n-1,1)} \oplus S^{(n-2,1,1)}$. In a moment, this space will be decomposed into components we already know about.

DEFINITION 4.6. We say that any procedure whose kernel contains the kernel of P (or, what is the same, its effective space is contained in the effective space of P) is *compatible with pairs*.

Any SRSF compatible with pairs will automatically send all complete head-to-head tie portions of the profile to zero. For $n = 3$, we saw that this meant it sent R and K to zero.

But which SRSFs are compatible with pairs? Certainly BC and KR are. But there are other schemes compatible with pairs (like the Condorcet, Simpson, Dodgson, and Copeland rules) which are not SRSFs; they are not even the same as 'composite ranking scoring functions', which take SRSFs and break ties with other SRSFs (see [**5**] for details on Copeland, for instance). The next example shows a simple example which *does* fall in this category.

EXAMPLE 4.7. In Figure 9 recall the rule from Example 3.10, now with sum-zero $\mathbf{s}$. This vector of weights has rotational *anti*-symmetry, but R_A has 180 degree

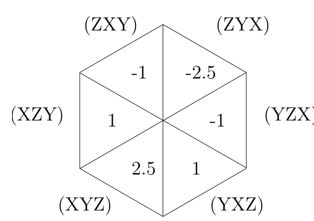

FIGURE 9. A procedure between Borda and Kemeny

rotational symmetry; hence, the procedure sends $R_A = (1, 1, -2, 1, 1, -2)$ to zero.

DEFINITION 4.8. We say that a neutral SRSF $f_{\mathbf{s}}$ *fully preserves pairs* if, as a linear transformation, it sends the subspace $S^{(n-1,1)} \oplus S^{(n-2,1,1)}$ from pairs compatibility to *exactly the same subspace.*

The subspace in the definition must be sent to an *isomorphic* subspace (or zero), by Schur's Lemma. However, in general it can be sent to *any* isomorphic component – for instance, the basis vector of a Borda component could be sent to some unusual linear combination of basis vectors of Alternating and Symmetric components. *But such components are themselves full pairwise ties.* So if a procedure fully preserves pairs, then the pairwise information from the Borda component is in some sense not 'wasted' – a key ingredient in the statement of Theorem 5.12.

EXAMPLE 4.9. There are rules which fully preserve pairs while not in fact being compatible with pairs, such as any positional scoring rule other than the Borda Count. Using plurality in Example 4.1, the *Sym* component was not killed;

still, the Borda component is sent to a scalar multiple of itself (and the Condorcet component to zero).

EXAMPLE 4.10. Using the same $\mathbf{s}$ as in Example 4.7, we can explicitly compute where B and C are sent. In fact, this SRSF happens to send C to itself, not even to a scalar multiple; KR *doubles* the influence of the Condorcet component. Similarly, the influence of B_X is multiplied by seven (by eight for BC and by six for KR).

In general, an $\mathbf{s}$ from the general continuum in Figure 7 sends B_X to $(2a+2)B_X$ and sends C to $(2a-4)C$.

5. Theorems and the Borda-Kemeny Spectrum

5.1. Statements of Theorems. We are now ready to state our results.

The first theorem is analogous to Theorem 5 in [**22**], and generalizes Theorem 6 of [**7**].

THEOREM 5.1. *The effective space of any neutral SRSF which is compatible with pairs is $S^{(n)} \oplus B \oplus C$, where the latter two are the Borda and Condorcet spaces defined above. Its image might lie in any piece of $S^{(n)} \oplus (n-1)S^{(n-1,1)} \oplus \binom{n-1}{2} S^{(n-2,1,1)}$, however.*

This can be proved most easily by noting that the KR and BC both kill complete head-to-head ties, but take B and C to multiples of themselves (hence B and C must be the specific copies of $S^{(n-1,1)}$ and $S^{(n-2,1,1)}$ in question).

PROPOSITION 5.2. *The BC sends any Borda component B_X to a multiple of B_X.*

PROPOSITION 5.3. *The KR sends any Condorcet component C_{XY} to a multiple of C_{XY}.*

Recall from Section 4.2 (see also Theorem 7.1) that the irreducible isomorphism classes $S^{(n-1,1),\pm}$ and $S^{(n-2,1,1),\pm}$ are P_n-modules, essentially differing in the same way that $S^{(2,1)}_{1,2}$ differed in Proposition 3.8. So if we look up the size of these pieces in the decomposition of Theorem 7.1 (in the Appendix) and note that $B \cong S^{(n-1,1),-}$ and $C \cong S^{(n-2,1,1),-}$ in that notation, we have:

THEOREM 5.4. *With reversal symmetry obeyed, however, the image of a neutral SRSF compatible with pairs must be in a space isomorphic to*
$S^{(n)} \oplus \frac{1}{2}\left(n-1+\left(\frac{1+(-1)^n}{2}\right)\right) S^{(n-1,1),-} \oplus \frac{1}{2}\left(\binom{n-1}{2} + \left\lfloor \frac{n-1}{2} \right\rfloor\right) S^{(n-2,1,1),-}$.

What about positional scoring rules? We state the SRSF analogue of the remark before Theorem 4 in [**7**] (the proof is essentially the same).

THEOREM 5.5. *The effective space of any SRSF which is a positional scoring rule (unless its vector of weights is sum-zero) is isomorphic to $S^{(n)} \oplus S^{(n-1,1)}$, where the $S^{(n-1,1)}$ component may be any piece of the whole $(n-1)S^{(n-1,1)}$ piece of $\mathbb{Q}S_n$.*

In fact, the copy of $S^{(n-1,1)}$ will depend on the structure of $\mathbf{s}$, as pointed out there. Nonetheless, the SRSF point of view is quite enlightening.

THEOREM 5.6. *With reversal symmetry obeyed, a positional scoring rule must have its effective space be $S^{(n)} \oplus S^{(n-1,1),-}$.*

This certainly makes sense, and this would also directly impact $\mathbf{s}$, as the weights $\mathbf{w}(i)+\mathbf{w}(n+1-i)$ must be invariant with respect to i in that case, or to equal zero if $S^{(n)}\to 0$.

In the next few more propositions, these techniques reprove that scoring rules using pairwise information must essentially be the Borda Count – a now-standard result.

Proposition 5.7. *Any profile orthogonal to $S^{(n)}\oplus(n-1)S^{(n-1,1)}$ has the property that it is sum-zero not just as a whole, but also for the subset of rankings r such that $r(j)=X$, for any j and any X.*

Proof. The structure of the $S^{(n-1,1)}$ components are such that each has a basis of vectors $\mathbf{p}$ such that $\mathbf{p}(v)$ is the same for all v with $v(j)=X$ (this follows immediately from the table in Section 4.2). Any vector which has value 1 for all $v(j)=X$ and zeros elsewhere is in the subspace given by the sum of the (one-dimensional) $S^{(n)}$ subspace and the right $S^{(n-1,1)}$ component. A profile orthogonal to this subspace must necessarily fulfill both requirements of the proposition. □

As a result, the score allocated to ranking r from any profile orthogonal to $S^{(n)}\oplus(n-1)S^{(n-1,1)}$ will be

$$\sum_{k=1}^{n}\left(\sum_{i=1}^{n-1}\left((n-i)\sum_{v(k)=X}t(v,r(i))\right)\right)$$

where the innermost sum is counted with (possibly negative) multiplicity, and hence must be zero by the proposition. This leads us to the generalization of what we discovered for plurality with $n=4$ in Example 4.1:

Proposition 5.8. *The image of any positional scoring rule will be in $S^{(n)}\oplus B$.*

Just as moving to the algebraic viewpoint gives us cardinal, not just ordinal, information, this gives us more information than before. Namely, positional scoring rules are extremely limited in their outcome potential; depending on your point of view, this might be good or bad. It certainly limits the types of ties one can have, for instance; it also means a lot of complete pairwise tie information (such as the *Sym* components!) is being interpreted dubiously.

Since the intersection of $(n-1)S^{(n-1,1)}$ and $B\oplus C$ is B, we now have the following, which extends Theorem 6 of [**7**] (itself an extension of various results of Saari) to the SRSF context.

Corollary 5.9. *An SRSF which is both a scoring rule* and *relies only on pairwise information has an effective space and image of $S^{(n)}\oplus B$. This must be essentially the same as the Borda count (or its reversal).*

We are now ready to state and prove the main theorem.

5.2. The Borda Count and the Kemeny Rule. Even if a neutral SRSF has lots of nice properties, there are still weird things that can happen, as the following two examples demonstrate.

Example 5.10. Assume $n=4$, and create the reversal-symmetry-obeying SRSF which sends B_X to Alt_X. Alt_X has -1 voters for each ranking with X first, $+1$ voters for each with X last, $+3$ for each ranking with X second, and

-3 for each ranking with X third. Given that B_X expresses very strong support for X, a procedure which interprets this as support for X in second place seems problematic.

EXAMPLE 5.11. Perhaps a procedure which kills off the Condorcet component *and* obeys reversal symmetry would be better. But with $n = 4$, one can construct just such an SRSF which sends B_A to $-40Alt_A$, going from overwhelming approval for A over all others to overwhelming approval for any profile with A in third place, some for ones with A in first, but the least for those with A in second! The vector $\mathbf{s}$ would have $s(ABCD, ABCD) = 0$, $s(BACD, ABCD) = -2$, but $s(ACBD, ABCD) = 3$, and $s(BDAC, ABCD) = -5$.

Given the vector of weights $\mathbf{s}$, this is not a procedure one would actually use – but that is not the point. Just as in Saari's papers [**18, 19**], the point is that *any* neutral, reversal-symmetric SRSF $f_{\mathbf{s}'}$ such that $\mathbf{s}'$ had a component of this $\mathbf{s}$ in it would incorporate some of that strange behavior.

Given the desire to avoid the behavior in the preceding examples, one must seriously consider the remaining alternatives; this is the essence of the algebraic point of view of voting theory. Once we have bothered to *get* a reasonable effective space of profiles by relying only on pairwise information, we will probably want to send that effective space to itself. This is the point of combining compatibility with pairs with the property of fully preserving pairs, and of the main theorem of the paper.

THEOREM 5.12. *Suppose a neutral SRSF is compatible with pairs* and *fully preserves pairs. Then (up to essential difference) this SRSF is on a one-dimensional continuum of procedures; this is precisely the continuum of procedures given by the Borda Count and Kemeny Rule.*

If in addition the Condorcet component goes to zero, the rule is the Borda Count.

We call this continuum the *Borda-Kemeny Spectrum.*

To prove Theorem 5.12 is mostly computation. By applying the additional hypotheses to Theorem 5.1 (which kills $S^{(n)}$), we see that such an SRSF must go from $B \oplus C$ to itself, hence the space is one-dimensional up to essential difference. To prove that the BC and KR are in fact on this continuum, simply collate Propositions 5.2 and 5.3, Corollary 5.9, and the following result.

PROPOSITION 5.13. *The KR sends any Borda component B_X to a multiple of B_X.*

5.3. The Borda-Kemeny Spectrum. Why might one be interested in such methods and procedures? Let's begin with some fairly concrete computations.

For convenience, we take $s(r, r) = 1$ and $s(r, \rho(r)) = -1$. For $n = 3$ the continuum with parameter t takes the shape[16] $(1, t, -t, -1, -t, 1)$, with $t = 1/3$ being Kemeny and $t = 1/2$ being Borda.

For $n = 4$ the continuum is more subtle. Assuming again that $s(r, r) = 1$ and $s(r, \rho(r)) = -1$, for a ranking r which is one neighbor swap away from v (as in the comment after Definition 2.3), we would have $s(v, r) = 2t$. For most r at distance two we would have t points, but $s(XYZW, YXWZ) = 4t - 1$. We would have

[16] This presentation differs slightly from before so it is easier to compare with $n = 4$.

$s(XYZW, YWXZ) = s(XYZW, ZXWY) = 0$, but the others at distance three having $\pm(3t-1)$, and those further away having negative points. The Kemeny Rule is at $t = 1/3$, the Borda Count at $t = 2/5$.

Notice that this spectrum already has more complexity; for instance, $4t - 1$ could be greater than or less than t depending on whether t was greater than $1/3$ or not. As a result, it is much more difficult combinatorially to obtain sharp outcomes; here is a useful one based on a weak Pareto-type condition.

Proposition 5.14. *If we assume that the partial order on the permutahedron is respected by* $\mathbf{s}$*, then we must have* $1/4 \leq t \leq 1/2$*.*

Example 5.15. What is particularly interesting about this is that there are reasonable (from this point of view) methods both 'between' KR and BC, but also on either side of them along the continuum! For instance, if $t = 1/4$, the partial order is respected but the C_{XY} Condorcet components are sent to (small scalar multiples of) $-C_{XY}$. This is an intriguing possibility if one wanted a procedure that intentionally *controverted* the expectations of cyclic profiles slightly.

Those who have studied voting methods from the linear-algebraic or geometric perspective have usually advocated for real-life use of the Borda Count based on its intense symmetry – especially since it takes cycles like $A \succ B \succ C \succ A$ and treats them as complete ties. For instance, [**7**] was motivated by trying to find analogues to BC for partial ranking information. For methods intended to provide a winner or set of winners, this seems reasonable to do, even if it might violate the Condorcet criterion.

Theorem 5.12 is significant because we now have a broader range of options for the Condorcet component in symmetric procedures. The Borda Count is the dividing line between Condorcet components being sent to themselves or their negatives, so one might reject social preference functions 'beyond' it (for $n = 4$, with $t < 2/5$) like the one in Example 5.15. But for procedures 'beyond' KR, it is not clear that there is an upper bound on how much influence should be given to the Condorcet components; one might want to approximate voting on a cyclic order itself. The end of Section 4 of [**35**] reports that when $n = 8$ the KR and BC give radically different outcomes; so the spectrum gains even more importance.

The spectrum (and these methods) should be useful in considering manipulation. It is 'classical' (originally due independently to Gibbard and Satterthwaite; see [**27**] for a comprehensive survey) that situations exist in nearly any choice system where a voter can cast a vote other than his or her actual preference and come out with a more preferred result. Geometric-algebraic methods have been of use in analyzing BC and KR (see for instance [**21**]); a taste of similar analysis is demonstrated next. (The proof is exceedingly tedious, but straightforward.)

Proposition 5.16. *Given a Borda-Kemeny spectrum method with vector of weights* $\mathbf{s} = (1, t, -t, -1, -t, t)$ *and a profile* $aB_A + bB_B + cC$ *(*a, b, c *constants), the precise border where manipulation can happen is not just when* $a = 2b$ *(between* ABC *and* BAC*), but also when* $b = -\frac{1-2t}{t+1}c$ *(between* ABC *and* ACB*) (The no-show paradox can occur when* $-\frac{1}{2} < 2a - 2b + c < 0$ *with KR.)*

Let us return to the ideas of Section 1.1. Choice theory is about *choice*, not just winners. In a situation of a board of directors, it is entirely reasonable for voters for $ABCD$ to say that they would prefer $BCAD$ to $ADCB$ as an outcome;

with $BCAD$ at least most of the succession is preserved, whereas with $ADCB$ their first-choice candidate wins but the rest of it is counter to their preference. We've already mentioned why it might be appropriate to leave a component of a profile which looks like $(1, 0, 1, 0, 1, 0)$ as a tie between the *rankings* ABC, BCA, CAB rather than a tie between *candidates* A, B, C.

There is one additional possible interpretation of this worth considering. The permutahedron is an abstract combinatorial object, but it may be embedded consistently in space in many ways. Zwicker has pointed out (in [**35**]; see also [**24**]) that one of the equivalent weighting vectors $\mathbf{s}$ for both KR and BC come from square distances between its vertices in different embeddings. Might there be a way to think of some of the other methods along this spectrum as part of the continuum stretching (for $n = 3$) the regular hexagon to the permutahedral vertices of the cube? (And if so, can we find a geometric interpretation of $t > 1/2$?) Ideally, this would give a natural connection to the representation theory as well – and the combinatorial structure has given us the tools.

6. Looking Forward

6.1. Algebra in the Service of Choice. Although it is a highlight, the Borda-Kemeny Spectrum is only part of the story; the SRSFs discussed are a good starting point, but not an end in themselves. Where might the types of thinking in this paper lead us?

Example 6.1. Suppose that the objects of voting really *are* cyclic orders; that is, each voter is allowed to select a cycle such as $A \succ B \succ D \succ C \succ A$. Natural contexts for this include:

- Seating preferences around a round table
- Rotating long-term site visit schedules for observation or inspection
- Scheduling space in a 24-hour facility

The combinatorial object which represents these in the same way the permutahedron represents rankings is called the *cyclic-order graph*[17]. With only three objects, it is nearly trivial (the only cyclic orders are $A \succ B \succ C \succ A$ and $A \succ C \succ B \succ A$, and these are equivalent if we add rotational/reversal symmetry). However, even with four 'candidates', there are six configurations – see Figure 10.

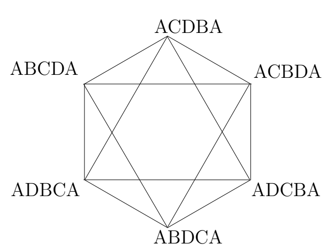

Figure 10. The cyclic order graph CO_4

This graph is the skeleton of the octahedron, which has symmetry group $S_4 \times C_2$ (for reasons unrelated to the permutahedron symmetry group). The 'profile space' is only six-dimensional, though, not even close to the size of the order of the

[17]See [**28**].

symmetry group. As a result, the irreducible components of the profile space are quite different.

One is the trivial component where all orders receive the same number of votes. There is also a two-dimensional one, roughly equivalent to the *Sym* component for SRSFs, which is generated by a profile differential with two votes for $ABCDA$ and its reversal, and -1 votes for each of the others. The third component is *three*-dimensional and is reminiscent of the Borda component – it is generated by profile differentials with one voter for a cyclic order, and minus one voter for its reversal.

If we focus on keeping only the 'Borda-like' component, we discover the space of possible procedures does *not* have symmetry around any square (4-cycle) in the graph. Instead, the weighting vectors look like $(a, b, c, -b, -c, -a)$, where the $-a$ is at the reversal of the cyclic order for a.

This example exemplifies our point of view in this paper. It is not possible to justify the more obvious sum-zero vectors $(a, 0, 0, 0, 0, -a)$ without introducing additional arguments – just like we had to introduce compatibility with pairs to focus attention on the most interesting procedures. The voting theory informed the algebra.

At the same time, there is some real voting theory, not just algebra, waiting to be done! What are natural non-algebraic conditions for 'nice' voting systems on cyclic orders? How do people really choose to sit around a table? And does person X really care if person Y is at her right or left, as long as they are sitting close together? (In this context, it does matter – a different graph would ask what happens if it doesn't matter.) These are all questions that require input from the choice community – just as finding new, appropriate questions to 'ask' the Borda-Kemeny spectrum SRSFs will take some time.

6.2. Future Work and Acknowledgements. There are many opportunities for further work here.

- What are the natural generalizations of Pareto and unanimity in the context of the permutahedron, and what properties would they imply? (This is not obvious.)
- The continuum of procedures can be different from BC and KR – how different? To what extent do they share desirable properties – especially for $n \geq 5$ (see [**35**])?
- What about truncated, tied, or incomplete preferences in this context?
- Cyclic order graphs, let alone representations of their automorphism groups, have not been studied much beyond [**28**]; what can we learn about voting in this context?
- What about voting with respect to the symmetries of some *arbitrary* graph on a set of alternatives?
- Can one give an explicit geometric model for the spectrum, as toward the end of Section 5.2?

6.2.1. *Acknowledgements.* Before acknowledging humans, I wish to explicitly point out that mathematical software (I used Sage [**26**]) was essential to discovering these rather subtle patterns, particularly when it came to more than $n = 4$

candidates. As Archimedes pointed out[18], it is much easier to prove something once you know what to prove! Thanks also go to:

- The organizers of a session where a very early version of this work was presented.
- Bill Zwicker for pointing out the connection to the Kemeny Rule and the Borda Count, and for many valuable references and discussion.
- Mike Orrison for enthusiastic support and references.
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- Readers of preprints, the referee, and the editor for conscientious comments, structural ideas, and very helpful improvements in exposition.

7. Appendix

7.1. Representations of the Permutahedron. Our main goal for this section is stating and proving Theorem 7.1 about precise irreducible decompositions of $\mathbb{Q}X$. We also collate several results about the representation theory of P_n and S_n, providing proofs where these are not well-known. The book [**13**] is a canonical reference, but we echo the notation from the more closely related [**8**].

It is classical that the irreducible representations of S_n (over a field of characteristic zero) are classified by partitions λ of n, each called S^λ. For instance, for $n = 4$ there are precisely five, labeled $S^{(1,1,1,1)}$, $S^{(2,1,1)}$, $S^{(2,2)}$, $S^{(3,1)}$, and $S^{(4)}$. The regular representation $\mathbb{Q}S_n$ decomposes, as a S_n-module, as $\bigoplus_\lambda \dim(S^\lambda)S^\lambda$.

The regular representation of S_n is given by the action of S_n on the vector space $\mathbb{Q}X$, where X is the set of permutations of $\{1, 2, \ldots, n\}$. But considered as the set of vertices of the permutahedron, there will be a $P_n \cong S_n \times C_2$ action on X, giving $\mathbb{Q}X$ a P_n-module structure as well.

Since P_n has such a nice structure, we know (see e.g. [**1**], Example 15.2) that each S^λ will be isomorphic (as an S_n-module) to two different irreducible P_n-modules, which we will call $S^{\lambda,+}$ and $S^{\lambda,-}$ to indicate how ρ acts on them (namely, $\rho S^{\lambda,+} = S^{\lambda,+}$ but $\rho S^{\lambda,-} = -S^{\lambda,-}$).

It turns out that most of this decomposition of $\mathbb{Q}X$ lies in the kernel from the perspective of voting theory. The important pieces are the $S^{(n-1,1)}$ and $S^{(n-2,1,1)}$ components, which are the ones pairwise-respecting procedures and points-based procedures are affected by.

THEOREM 7.1. *For $n > 3$, the decomposition of $\mathbb{Q}X$ as a P_n-module includes exactly the following number of copies of these irreducible submodules:*

Irreducible	*Number*
$S^{(n-1,1),+}$	$\frac{1}{2}\left(n - 1 - \left(\frac{1+(-1)^n}{2}\right)\right)$
$S^{(n-1,1),-}$	$\frac{1}{2}\left(n - 1 + \left(\frac{1+(-1)^n}{2}\right)\right)$
$S^{(n-2,1,1),+}$	$\frac{1}{2}\left(\binom{n-1}{2} - \left\lfloor\frac{n-1}{2}\right\rfloor\right)$
$S^{(n-2,1,1),-}$	$\frac{1}{2}\left(\binom{n-1}{2} + \left\lfloor\frac{n-1}{2}\right\rfloor\right)$

This result is allf the theorems need; these numbers are given for small n in the following table.

[18]With respect to both Democritus and Eudoxus deserving credit for showing that a cone or pyramid has one-third the volume of the respective cylinder, see e.g. [**11**] for discussion.

n	$\#S^{(n-1,1),+}$	$\#S^{(n-1,1),-}$	$\#S^{(n-2,1,1),+}$	$\#S^{(n-2,1,1),-}$
3	1	1	0	1
4	1	2	1	2
5	2	2	2	4
6	2	3	4	6
7	3	3	6	9
8	3	4	9	12
9	4	4	12	16

Hence the decomposition of $M^{(n-1,1)}$ is into $\frac{n-1}{2}$ Symmetric components and $\frac{n-1}{2}$ Borda/Alternating components when n is odd, and $\frac{n}{2}-1$ and $\frac{n}{2}$ when n is even. And the decomposition of $M^{(n-1,1,1)}$ similarly has $\binom{n-1}{2}$ total components with $\left\lfloor\frac{n-1}{2}\right\rfloor$ more of the 'minus' component.

Let π be the character of the P_n-module $\mathbb{Q}X$; these steps prove Theorem 7.1.

- Compute π.
- Compute the inner product (π,χ) on the space of class functions for a general character χ of an irreducible P_n-module.
- Compute (π,χ) for the specific χ we care about for the theorem.
- Apply those computations to the size of $S^{(n-1,1)}$ and $S^{(n-2,1,1)}$ to get the theorem.

For a given $g\in P_n$, the value $\pi(g)$ of the character π of $\mathbb{Q}X$ is the number of fixed points of X under that element (conjugacy class) of P_n ([**1**], Example 15.4). We write a generic element g as either $g=(\sigma,e)$ or $g=(\sigma,\rho)$, where $\sigma\in S_n$ and ρ is the reversing element mentioned earlier.

All vertices are fixed under the identity, so $\pi(e,e)=n!$. Since the action of a group on itself is transitive, $\pi(\sigma,e)=0$ if $\sigma\neq e$.

For the action of the other elements of P_n, we will look more closely at what is going on. Pick an arbitrary vertex p of the permutahedron; for the purposes of the action (left or right), this should be thought of as a permutation of the set $\{1,2,\ldots,n\}$. For p to be a fixed point for $g=(\sigma,\rho)$, it must be the case that (as permutations) $p=\rho p\sigma$. That is, for each $1\leq i\leq n$, we must have that $p(i)=n+1-p(\sigma(i))$, or $p(\sigma(i))=n+1-p(i)$. But then $p(\sigma(i))=n+1-p(\sigma(\sigma(i)))$ as well, so that $p(i)=p(\sigma(\sigma(i)))$, which by transitivity means $\sigma(\sigma(i))=i$ for all i, which means σ has order two.

This narrows σ down to permutations made up of disjoint transposes (j,k). Further, since $p(i)+p(\sigma(i))=n+1$, if $\sigma(i)=i$ for some i, then $p(i)=\frac{n+1}{2}$, and there can be only one such i. Hence σ is a permutation made up of as many disjoint transposes as possible, which is the cycle decomposition type of ρ; since the cycle decomposition type determines the conjugacy class of a permutation, σ must be in the conjugacy class of ρ! Otherwise there are no fixed points at all.

To simplify the computation if there are, assume $\sigma=\rho$. Then any p which has $p(i)+p(n+1-i)=n+1$ for all i will work. Once we have chosen $p(i)$ for $1\leq i\leq\left\lfloor\frac{n}{2}\right\rfloor$, that fixes the others. We can choose $p(1)$ to be anything except $\frac{n+1}{2}$ (if that is an integer), which is $2\left\lfloor\frac{n}{2}\right\rfloor$ choices, and which then removes $p(n)$ from consideration; then $p(2)$ can be any of the remaining $2\left(\left\lfloor\frac{n}{2}\right\rfloor-1\right)$ choices, and so on. Thus the number of fixed points for $g=(\rho,\rho)$ is $2^{\left\lfloor\frac{n}{2}\right\rfloor}\left(\left\lfloor\frac{n}{2}\right\rfloor!\right)$.

To summarize, $\pi(e,e)=n!$, $\pi(\rho,\rho)=\left(\lfloor\frac{n}{2}\rfloor!\right)2^{\lfloor\frac{n}{2}\rfloor}$, and $\pi(g)=0$ for all other elements of the group. Let $f(g)$ be the size of the conjugacy class of g. The conjugacy class of the identity is always just itself, while the conjugacy class of (ρ,ρ) is the set of all (σ,ρ) where σ have the same cycle type as ρ. The easiest way to think of such a σ is as a permutation which then has parentheses every two entries, yielding $\lfloor\frac{n}{2}\rfloor$ pairs; then we divide by the number of symmetries, of which there are 2 for each pair, and then divide by the permutations of the pairs.

Now decomposing the character π with respect to any irreducible character χ can be done directly:

$$(\pi,\chi)=\frac{1}{2\cdot n!}\left(\pi(e,e)\cdot f(e,e)\cdot\chi(e,e)+\pi(\rho,\rho)\cdot f(\rho,\rho)\cdot\chi(\rho,\rho)+0\right)=$$

$$\frac{1}{2\cdot n!}\left(n!\cdot\chi(e,e)+\left(\left\lfloor\frac{n}{2}\right\rfloor!\right)2^{\lfloor\frac{n}{2}\rfloor}\cdot\frac{n!}{\left(\lfloor\frac{n}{2}\rfloor!\right)2^{\lfloor\frac{n}{2}\rfloor}}\chi(\rho,\rho)\right)=\frac{1}{2}\left(\chi(e,e)+\chi(\rho,\rho)\right)$$

The following two propositions are enough to prove the voting assertions.

Proposition 7.2. *If $\chi=\chi_{S^{(n-1,1),-}}$, then $\chi(\rho,\rho)=\left(\frac{1+(-1)^n}{2}\right)$, which is to say it alternates between 0 and 1 for n odd and even.*

Proposition 7.3. *If $\chi=\chi_{S^{(n-2,1,1),-}}$, then $\chi(\rho,\rho)=\lfloor\frac{n-1}{2}\rfloor$, which is to say it goes through positive integers in order and repeats each value twice, once for n odd and once for n even.*

Before proving these statements, we finish the proof of Theorem 7.1. We already know that $\chi_{S^{(n-1,1),\pm}}(e,e)=n-1$ and $\chi_{S^{(n-2,1,1),\pm}}(e,e)=\binom{n-1}{2}$. For the $+$ components, the theorem is immediate. For same calculation with the $-$ components, it suffices to recall that $\chi_{S^{\lambda,-}}(\sigma,\rho)=-\chi_{S^{\lambda,+}}(\sigma,\rho)$.

Proof of Proposition 7.2. We look at the Borda component as being a typical example of $S^{(n-1,1),-}$. We know that $\chi(\rho,\rho)$ is the trace of the matrix given by the action of ρ on the right *and* the left of the permutahedron. We use the usual basis of $B_{A_1},\cdots,B_{A_{n-1}}$.

Conjugation by ρ is the 'swap' automorphism. It turns out that this sends a ranking with A_j in the ith position to one with A_{n+1-j} in the $(n+1-i)$-th position, as we noted when calculating fixed points, where $q(i)=n+1-p(n+1-i)$. But this automatically means that B_{A_i} is sent under this action to $-B_{A_{n+1-i}}$. Combining this with the fact that $-B_{A_n}=\sum_{i\neq n}B_{A_i}$, that means the matrix looks like

$$\begin{pmatrix}1&0&0&\dots&0&0\\1&0&0&\dots&0&-1\\1&0&0&\dots&-1&0\\ \dots&\dots&\dots&\dots&\dots&\dots\\1&0&-1&\dots&0&0\\1&-1&0&\dots&0&0\end{pmatrix}$$

which will clearly have the correct trace. □

Proof of Proposition 7.3. We look here at the (new) C_{XY} as a typical example of $S^{(n-2,1,1),-}$. We will use the basis mentioned above,

$$C_{A_1A_2},C_{A_1A_3},\cdots,C_{A_1A_{n-1}},C_{A_2A_3},\cdots,C_{A_2A_{n-1}},\cdots,C_{A_{n-2}A_{n-1}},$$

and once again will look at swapping as the action. Using the same argument as above, we see that if $A_i \succ A_j$ originally, then after swapping it is the case that $A_{n+1-j} \succ A_{n+1-i}$, so that $C_{A_iA_j} \to C_{A_{n+1-j}A_{n+1-i}}$.

The images of the basis are nearly all (different elements) in the basis too, not contributing to the trace, since our basis involves A_i, A_j where $i < j$, so $n+1-j < n+1-i$ as well. The only outlier case is when $n+1-i = n$, in which case $i = 1$, which only will contribute to the trace is if $i = n+1-j$ and $j = n+1-i$ (or $i+j = n+1$), or possibly if $i = 1$. Let's analyze this case.

When $i+j = n+1$, it contributes one to the trace. But for $0 < i < j < n$, the only pairs are for $i+j = n+1$ with $i > 1$, which means we only have to count these. So for odd n we get get one pair for each integer $2 \leq i < \frac{n}{2}$, which leaves $\left\lfloor \frac{n}{2} \right\rfloor - 1$. When $i = 1$, we need to get $C_{A_{n+1-j}A_n}$ in terms of the basis, which can be rewritten as

$$-\sum_{k \neq n+1-j,n} C_{A_{n+1-j}A_k} = -\sum_{n+1-j<k<n} C_{A_{n+1-j}A_k} + \sum_{0<k<n+1-j} C_{A_kA_{n+1-j}}\,.$$

This sum contributes to the trace precisely if there is a $C_{A_1A_j}$ as one of the terms, which can only happen if $n+1-j = 1$ and $k = j$, or if $k = 1$ and $n+1-j = j$. The first implies that $j = n$, which was not one of the original basis elements, but the second option implies that $j = \frac{n+1}{2}$. So if n is odd we must add one more.

Thus we arrive at a total trace of

$$\mathrm{Tr}(\text{conj. by } \rho) = \begin{cases} \frac{n}{2} - 1 = \left\lfloor \frac{n-1}{2} \right\rfloor, & n \text{ even} \\ \left\lfloor \frac{n}{2} \right\rfloor - 1 + 1 = \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n-1}{2} \right\rfloor, & n \text{ odd} \end{cases} = \left\lfloor \frac{n-1}{2} \right\rfloor.$$

□

7.2. Proofs of Preference Function Properties.

Proof of Proposition 5.2. Consider that B_X consists of a profile with $n+1-2k$ voters for each ranking with X in kth place. Recall that the SRSF gives $\sum_v s(v,r)$ points to ranking r, where the sum is over the whole profile (in the case of a differential, we can just subtract the negative v). We will see that the Borda Count sends B_X to a multiple of B_X, which by symmetry will mean that this is the component.

Since BC is a positional scoring rule, for a given r we have that $s(v,r) = (n-1)t(v,r(1)) + (n-2)t(v,r(2)) + \cdots + t(v,r(n-1))$. When we break up the sum over all v into sums over each of the subsets where $v(k) = X$, we get a double sum

$$\sum_{k=1}^{n} \left(\sum_{v(k)=X} (n+1-2k)\left[(n-1)t(v,r(1)) + (n-2)t(v,r(2)) + \cdots + t(v,r(n-1))\right] \right)$$

which exhibits a very high degree of symmetry.

Assume that $X = r(j)$. Consider the two terms in the double sum above for some k and the corresponding $n+1-k$ (which will have opposite sign). Since B_X has the same number of voters for *all* rankings with $v(k) = X$, then for a given $v(i) = r(\ell)$, the two sets

$$\{v | v(k) = X, v(i) = r(\ell), i \neq k, n+1-k\} \text{ and}$$
$$\{v | v(n+1-k) = X, v(i) = r(\ell), i \neq k, n+1-k\}$$

have the same size. That means that the terms in $s(v,r)$ corresponding to these sets will cancel, since they correspond to $(n-\ell)t(v,r(\ell))$ when $r(\ell)=v(i)$, and there are equal numbers of these for k and $n+1-k$ as long as $i\neq k, n+1-k$.

So, for each v such that $v(k)=r(j)=X$ and $v(n+1-k)=r(i)$ (where obviously $i\neq j$), the non-canceling part of the term is

$$(n+1-2k)\left[(n-j)t(v,r(j))+(n-i)t(v,r(i))\right],$$

which is okay since when $k=1$ we correctly get 0 as the inside coefficient in $s(v,r)$. Sum this back up and substitute in the Borda Count values of $t(v,r(j))=t(v,X)=\frac{n-k}{n-1}$ and $t(v,r(i))=\frac{n-(n+1-k)}{n-1}=\frac{k-1}{n-1}$ to get

$$\sum_{k=1}^{n}\left(\sum_{v(k)=X=r(j),\, v(n+1-k)=r(i)}(n+1-2k)\left[(n-j)\frac{n-k}{n-1}+(n-i)\frac{k-1}{n-1}\right]\right).$$

There are $(n-1)!$ different v such that $v(k)=X$, and hence $(n-2)!$ different v such that $v(k)=X$ *and* $v(n+1-k)=r(i)$ in the above sum. Then we get

$$\sum_{k=1}^{n}(n+1-2k)\left((n-1)!\,(n-j)\frac{n-k}{n-1}+(n-2)!\sum_{i\neq j}(n-i)\frac{k-1}{n-1}\right).$$

In fact, a little clearing of denominators yields

$$(n-2)!\sum_{k=1}^{n}(n+1-2k)\left((n-j)(n-k)+\sum_{i\neq j}(n-i)\frac{k-1}{n-1}\right).$$

The reader will notice that the sum only depends on j, as one would hope. If we increase j by one, the difference between two of these scores is

$$(n-2)!\sum_{k=1}^{n}(n+1-2k)\left((n-j)(n-k)+\sum_{i\neq j}(n-i)\frac{k-1}{n-1}\right)-$$
$$(n-2)!\sum_{k=1}^{n}(n+1-2k)\left((n-(j+1))(n-k)+\sum_{i\neq j+1}(n-i)\frac{k-1}{n-1}\right)$$

which can be simplified to a formula *not* depending on j, as needed:

$$(n-2)!\sum_{k=1}^{n}(n+1-2k)\left((n-k)-\frac{k-1}{n-1}\right)=(n-2)!\frac{n^2(n+1)}{6}.$$

Similarly, we need that $j=1$ and $j=n$ are opposites, and indeed

$$(n-2)!\sum_{k=1}^{n}(n+1-2k)\left((n-1)(n-k)+\sum_{i\neq 1}(n-i)\frac{k-1}{n-1}\right)+$$
$$(n-2)!\sum_{k=1}^{n}(n+1-2k)\left((n-n)(n-k)+\sum_{i\neq n}(n-i)\frac{k-1}{n-1}\right)$$

simplifies to

$$\begin{aligned}
&(n-2)!\sum_{k=1}^{n}(n+1-2k)\left((n-1)(n-k)+\frac{k-1}{n-1}\left((n-1)+(n-n)+2\sum_{i=2}^{n-1}n-i\right)\right)\\
=&(n-2)!\sum_{k=1}^{n}(n+1-2k)\left((n-1)(n-k)+\frac{k-1}{n-1}\left(\frac{n(n-1)}{2}+\frac{(n-1)(n-2)}{2}\right)\right)\\
=&(n-2)!(n-1)^2\sum_{k=1}^{n}(n+1-2k)=0\,.
\end{aligned}$$

□

Proof of Proposition 5.3. For the Kemeny Rule, recall that the SRSF function is given by $s(v,r)=\sum_{a,b\in A}\delta(v,r,a,b)$, where δ is 1 if $a\succ b$ by both rankings v and r, and is 0 otherwise; clearly this relies only on pairwise information. Let us see where it sends the (new) Condorcet components.

Using the notation above for $\{XYi\}$ we see that for a given ranking r (with $r\in\{XYj\}$) the score for r is

$$\sum_{i=1}^{n-1}(n-2i)\sum_{v\in\{XYi\}}\sum_{a,b\in A}\delta(v,r,a,b)\,.$$

This depends only on j since the sums are over all $v\in\{XYi\}$. For each $v\in\{XYi\}$ such that $a\succ_v b$ and $a\succ_r b$, reversing r will cause δ to go from 1 to 0, but will send δ from 0 to 1 for the reversal of v; since this reversal is in $\{XY(n-i)\}$, the score for the reversal of r is

$$\sum_{i=1}^{n-1}(n-2i)\sum_{v\in\{XY(n-i)\}}\sum_{a,b\in A}\delta(v,r,a,b)=\sum_{k=1}^{n-1}(2k-n)\sum_{v\in\{XYk\}}\sum_{a,b\in A}\delta(v,r,a,b)$$

which is a change of sign, as expected.

Furthermore, if r changes from $\{XYj\}$ to $\{XY(j+1)\}$ via a one-position swap, then all δ values will be the same except ones concerning that pair. So each v with that pair as in r loses 1, while each with the pair as in r' gains one.

If such a swap changes things, it must involve X or Y; we will suppose it is $Y\_$ changing to $\_Y$ (other possibilities are very similar). The number of potential places for the candidate $\_$ to occur *after* Y in a ranking in the set $\{XYi\}$ varies from zero to $n-2$, but there will be two potential ones of type $n-1-i$ (such as, for $i=2$, $X\_Y\_\dots\_$ and $\_Y\_\dots\_X$). Hence there are, for $\{XYi\}$, $n-1-i+\sum_{i=0}^{n-2}i=n-1-i+\frac{(n-1)(n-2)}{2}$ possibilities out of a total of $n(n-2)$ total; the difference is

$$n-1-i+\frac{(n-1)(n-2)}{2}-\left(n(n-2)-\left(n-1-i+\frac{(n-1)(n-2)}{2}\right)\right)=$$

$$2\left(n-1-i+\frac{(n-1)(n-2)}{2}\right)-n^2+2n=2n-2-2i+(n-1)(n-2)-n^2+2n=n-2i$$

The other $n-3$ spots have $(n-3)!$ different possibilities, so there are $(n-2i)(n-3)!$ net rankings v in $\{XYi\}$ which will lose $\delta=1$ (which, for i for which this is negative, correspond to gaining $\delta = 1$). Thus the difference in the scores

$$\sum_{i=1}^{n-1}(n-2i)\sum_{v\in\{XYi\}}\sum_{a,b\in A}\delta(v,r,a,b)$$

will be

$$\sum_{i=1}^{n-1}(n-2i)\left[(n-2i)(n-3)!\right]=(n-3)!\sum_{i=1}^{n-1}(n-2i)^2=\frac{n!}{3}$$

□

Proof of Proposition 5.13. We use the same decomposition as above for the Borda Count, over $v(k) = X = r(j)$. As above,

$$\sum_{k=1}^{n}\left(\sum_{v(k)=X}(n+1-2k)\sum_{a,b\in A}\delta(v,r,a,b)\right).$$

This also only depends on j (here, r is a fixed ranking with $r(j) = X$, but nothing else known) because of the sum over all v with each $v(k) = X$. However, it is useful to focus on a specific r for proving this sends Borda to Borda.

First let us observe what happens to the score when r is reversed. For each v such that $a \succ_v b$ and $a \succ_r b$, we will get $+(n + 1 - 2k)$, depending on $v(k) = X$. But if r is reversed, these go away, and the reversal of v will have $\delta = 1$. This will exactly give the negative of the original score, because if $v(k) = X$, then the reversal v' has $v'(n+1-k) = X$, which means it will contribute $+(n+1-2(n+1-k)) = -n-1+k = -(n+1-k)$ to the score.

We also need to have a fixed change in score when r changes. Let r' be the same ranking as r but with $r'(j + 1) = X$. As with the C_{XY} components, we will suppose the change is $X\_$ changing to $\_X$. The number of potential places for the candidate _ to occur *after* X in a ranking in the set $\{v(i) = X\}$ is $n - i$, and the number of places before is $i - 1$, so the difference is $n + 1 - 2i$. For the remaining spots there are $(n - 2)!$ possibilities, so there are $(n + 1 - 2i)(n - 2)!$ net rankings v in $\{v(i) = X\}$ which will lose $\delta = 1$ (or, if $(n + 1 - 2i)(n - 2)! < 0$, gain $\delta = 1$).

Thus the difference in the scores given by

$$\sum_{k=1}^{n}\left(\sum_{v(k)=X}(n+1-2k)\sum_{a,b\in A}\delta(v,r,a,b)\right)$$

will be

$$\sum_{k=1}^{n}\left((n+1-2k)^2(n-2)!\right)=(n-2)!\sum_{k=1}^{n}(n+1-2k)^2=\frac{(n+1)!}{3}$$

□

Proof of Proposition 5.8. Let us see what happens to the profile $\mathbf{p}$ which has value $\mathbf{p}(v) = 1$ for $v(j) = X$ and zeros otherwise. In this case,

$$\sum_{k=1}^{n}\left(\sum_{v(k)=X}\left(\sum_{i=1}^{n-1}(n-i)t(v,r(i))\right)\right)=\sum_{v(j)=X}\left(\sum_{i=1}^{n-1}(n-i)t(v,r(i))\right)$$

Now suppose that $r(\ell) = X$; then it makes sense to rewrite this as

$$\sum_{v(j)=X} \left((n-\ell)t(v, r(\ell)) + \sum_{i=1,\, i\neq \ell}^{n-1} (n-i)t(v, r(i)) \right) .$$

Call the weighting vector w. Then note that in the second sum inside the parentheses, for a given i, $v(k) = r(i)$ the same number of times (to be precise, $(n-2)!$ times), other than $v(j)$, of course. So we can rewrite this

$$\sum_{v(j)=X} \left((n-\ell)w(j) + \sum_{v(j)=X} \sum_{i=1,\, i\neq \ell}^{n-1} (n-i)t(v, r(i)) \right) =$$

$$(n-1)!(n-\ell)w(j) + \sum_{i=1,\, i\neq \ell}^{n-1} (n-i)(n-2)! \sum_{k=1,\, k\neq j}^{n} w(k) =$$

$$(n-2)! \left[(n-1)(n-\ell)w(j) + \sum_{k=1,\, k\neq j}^{n} \sum_{i=1,\, i\neq \ell}^{n-1} w(k)(n-i) \right]$$

It is easy to see that the difference in this caused by changing $r(\ell) = X$ to $r(\ell+1) = X$ is $(n-2)! \left[(n-1)w(j) - \sum_{k=1,\, k\neq j}^{n} w(k) \right]$, which does not depend on ℓ. Likewise, adding the cases for $\ell = 1$ and $\ell = n$ gives

$$(n-2)! \left((n-1)^2 w(j) + \sum_{k=1,\, k\neq j}^{n} w(k) \left(\sum_{i=1}^{n-1} n-i + \sum_{i=2}^{n-1} n-i \right) \right) =$$

$$(n-1)!(n-1) \sum_{k=1}^{n} w(k)$$

so that it goes to the Borda component alone if the weighting vector is sum-zero, otherwise it is 'shifted' by a multiple of the sum of the weights. □

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Gordon College, 255 Grapevine Road,, Wenham, MA 01984
E-mail address: karl.crisman@gordon.edu