THE BORDA COUNT, THE KEMENY RULE, AND THE PERMUTAHEDRON

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Abstract. A strict ranking of n items may profitably be viewed as a permutation of the objects. In particular, social preference functions may be viewed as having both input and output be such rankings (or possibly ties among several such rankings). A natural combinatorial object for studying such functions is the permutahedron, because pairwise comparisons are viewed as particularly important.

In this paper, we use the representation theory of the symmetry group of the permutahedron to analyze a large class of such functions. Our most important result characterizes the Borda Count and the Kemeny Rule as members of a highly symmetric one-parameter family of social preference functions.

1. Introduction

A big theme in the mathematics of voting and choice is symmetry in some setting where preferences are aggregated. Since individual preferences may be represented as combinatorial objects, the symmetries of such objects are largely what is at issue (as well as real-life applications thereof). Indeed, though they do not use the same formal language, all the foundational papers (e.g., [2, 3, 5, 11]) constantly refer to this[1]; today, many such notions have been standardized (see below).

For concreteness, consider an election among n candidates, $A_1, A_2, \ldots, A_n$; we usually represent any individual voter's preferences as a strict transitive ranking such as $A_1 \succ A_3 \succ \cdots \succ A_2$. Hence, one might think of a ranking as a permutation of the set of candidates (or even just of the set $\{1, 2, \cdots, n\}$), and hope that information about permutations yields information about different aggregation procedures. For another example, coalitions on up/down votes (ignoring abstentions to illustrate) in the United Nations Security Council (an instance of cooperative game theory) are simply subsets of the power set of the set of voters, which has a natural combinatorial structure.

In fact, it turns out that many of the various 'natural' fairness requirements in social choice can be thought of this way - particularly, though not exclusively, in terms of the symmetric group. If the procedure is invariant under some group action, we could think of that as being a more equitable procedure. For instance, when discussing voting procedures, symmetry under the action of the symmetric group on n candidates on $\{A_i\}_{i=1}^n$ (denoted throughout by S_n) is usually known as *neutrality*; this essentially means that no candidate has an unfair advantage in the

[1] For instance, in [3] it is crucial that every possible set of preferences be in the domain of their functions; in [5], a major assumption is that voters' preferences of subsets of candidates obey various (anti-)symmetric partial orders.

election[2]. This same action works on the set of voters in the game theory context, though the Security Council does not have this symmetry due to the nuclear powers having veto power.

Although these kinds of symmetry have been part of social choice theory from its inception, it is only relatively recently that it has moved from an axiomatic approach, which tried to narrow many choices of rules to a single one using such 'fairness' criteria, to a classification approach, where aggregation procedures are viewed within a large range of rules as to how much they might obey some quantifiable symmetry. There are many examples of this, but particularly relevant to this study are the three pairs of articles [18, 19], [22, 26], and [31, 32].

Within the last five years, there have been several moves to go further and use the representation theory of the symmetric group to classify – usually with a view to first reframe previous results, then extend them in a more general framework. Orrison and his students ([8]) have done so in voting theory (largely with a view to utilizing incomplete preferences), while recent work of Hernández-Lamoneda, Juárez, and Sánchez-Sánchez ([13]) gives similar results in cooperative game theory. More recently, similar techniques have been used in joint work of Bargagliotti and Orrison in nonparametric statistics ([4]). One great advantage of this has been to generalize known results without the technical challenges long associated with this; just as important is the deeper insight gained into *why* the results are true.

But there are combinatorial structures other than permutations which play a role in fairness, and other groups than the symmetric group to consider. Pairwise comparisons between candidates have been a cornerstone of analysis since [2], for instance. One may also note that a ranking of candidates is not simply a permutation, but an ordering.

With this in mind, this paper will focus on the role played by the permutahedron, as the natural object to keep track of rankings in context. We will use its symmetry group to shed light on the structure of a fairly general class of voting rules due to Conitzer and Zwicker (see section 2.1 below), called *simple ranking scoring functions* (SRSFs). The 'extra' symmetry of the permutahedron will correspond precisely to a well-known voting theory concept called *reversal* symmetry (see section 2.3).

The most important result we obtain from this classification is a one-parameter family of such rules connecting two well-known procedures. We will fully define these below, but intuitively one may think of the Borda Count as being a voting rule which gives k points to the kth candidate from the bottom in a given voter's ranking, while the Kemeny Rule is similarly regular but gives points to *rankings* based on a certain implied 'distance' from a given voter's ranking.

Main Theorem. *The Borda Count and Kemeny Rule are both members of a one-parameter family of voting procedures which is highly symmetric with respect to pairwise information and the permutahedron.*

See below (Theorem 23) for a full statement. In fact, by adding one final symmetry, one can characterize the Borda Count among SRSFs in the same way as is usually done with it among positional scoring rules or rules relying only on pairwise information.

[2]This is not true of many systems in use, such as in a party primary system; considering the *whole* election cycle, a candidate in an uncontested primary has an advantage toward winning the whole thing over one who has a primary battle she may or may not win. This may not always be true *politically*, but can be demonstrated *mathematically* when all else is equal (as it rarely is).

Our tool to analyze this group of social choice procedures is the basic representation theory of the permutahedron (summarized in the Appendix). This allows us to classify things such as positional scoring rules among SRSFs in terms of subspaces of the space of all possible sets of voters (called *profiles*, see below), rather than more axiomatic or calculational terms. See the introduction to section 4 for general statements and references to precise statements later in that section.

Most proofs will be postponed to the Appendix. We will also give occasional examples of how these techniques can facilitate investigating more qualitative choice criteria, such as consistency or Pareto conditions. However, the thrust of the paper is to demonstrate how the algebra gives insight, not to address SRSFs comprehensively, so we will not go in depth with such discussions.

The outline of the remainder of the paper is as follows:

- Review social choice definitions
- Review connections to representation theory, with explicit examples for $n = 3$
- State and prove theorems for all n

Finally, we wish to again emphasize that the permutahedron is the 'right tool' for this job, and look forward to seeing future work in social choice and voting using a wide range of discrete structures.

2. Voting and Social Choice

2.1. **General Definitions.**

We begin by recalling relevant general definitions from voting theory; we use notation from the various relevant articles and books ([25, 6, 16, 17, 27]) as appropriate.

Let $A = \{A_1, A_2, \ldots, A_n\}$ be a set of n candidates/alternatives; we may refer to generic ones by capital letters at the beginning (A, B, C) or end (X, Y, Z) of the alphabet. Let $L(A)$ be the set of (strict) linear rankings of those alternatives, such as $A_1 \succ A_2 \succ \cdots \succ A_n$. Obviously these orders correspond to permutations of A, and we will often identify $L(A)$ and S_n by abuse of notation; it should always be clear whether we are referring to a ranking or a permutation. Given a ranking r, if $X \succ Y$ in the order implied by r, we say that $X \succ_r Y$. Likewise, for any $1 \leq i \leq n$ we denote the ith ranked alternative in r by $r(i)$.

A *profile* $\mathbf{p}$ is simply a vector-valued function $\mathbf{p} : L(A) \to \mathbb{Q}$, where one interprets each value as the number of voters[3] who prefer a given ranking in $L(A)$. Because of this interpretation, we will often use the notation $\sum_{v\in\mathbf{p}} \mathbf{p}(v)f(v)$ to denote evaluating some function f over each ranking $v \in L(A)$ with multiplicity $\mathbf{p}(v)$, the 'number of voters preferring v in $\mathbf{p}$'.

A *social preference function* is a function from the set of all (finite) profiles to the power set of $L(A)$ (excepting the empty set). One might think of a social preference function as taking an electorate's set of preferences and yielding some nonempty set of rankings. (Note that these output rankings might not necessarily be related to each other, though they often will be.)

One may get a social preference function from a social welfare function (where the output is a single weak linear ranking of alternatives) or from a voting rule (where the output is a nonempty set of alternatives) by taking the set of all compatible

[3]It is by now common practice to do this so it is a vector space; it also enables easy normalization, if desired.

rankings, so we see that this extends more familiar notions appropriately. The key example of this is the following.

Definition 1. Let a *weighting vector* be an arbitrary[4] $\mathbf{w} \in \mathbb{R}^n$. Given a profile $\mathbf{p}$, we say an alternative X receives the *score* $\sum_{i=1}^{n} \sum_{v \in \mathbf{p},\, v(i)=X} \mathbf{p}(v)\mathbf{w}(i)$ from the weighting vector; the set of alternatives with the maximum such score is the set of *winners*. The *positional scoring rule* associated to w is the social preference function which associates to each profile $\mathbf{p}$ all rankings r which are compatible with all winners being ranked above all non-winners.

Even in this preference function garb, positional scoring rules are quite familiar. If $\mathbf{w} = (1, 0, 0, \cdots, 0)$, then we have the usual plurality vote used in most elections in the United States ('vote for your favorite') and if $\mathbf{w} = (n-1, n-2, \ldots, 1, 0)$ it is the *Borda Count* (BC), which assigns $n - i$ points to the ith place alternative for a given voter[5].

2.2. **Neutral Simple Ranking Scoring Functions.** Our main object of study are functions which are essentially scoring rules, but with scores given to full rankings rather than individual candidates. The following definition is due to Conitzer et al., though it is very similar to a roughly contemporaneous definition of generalized scoring rules in Zwicker (in the case $I = O = L(A)$)[6]. See [6] and [32].

Definition 2. A social preference function f is a *simple ranking scoring function* (SRSF) if there exists a function $s : L(A) \times L(A) \to \mathbb{R}$ such that for all votes v, $f(v)$ is the ranking(s) r which maximize $\sum_{v \in \mathbf{p}} \mathbf{p}(v)s(v, r)$. It is *neutral* if s is neutral, that is if for any $\sigma \in S_A$, $s(v, r) = s(\sigma(v), \sigma(r))$. (In particular, this means that the permutation sending v to r must work, so that $s(v, r) = s(r, v)$.)

The power of the SRSF concept comes from it being a common generalization of two otherwise disparate types of systems.

On the one hand, one often-studied (though less often used in practice) rule is very naturally defined as a neutral SRSF, as in Proposition 2 of [6].

Definition 3. Let v and r be as above, and $a, b \in A$; then define the function

$$\delta(v, r, a, b) = \begin{cases} 1, & a \succ_r b \text{ and } a \succ_v b \\ 0, & \text{otherwise} \end{cases}.$$

The *Kemeny Rule* (KR) is the neutral SRSF such that $s(v, r) = \sum_{a,b \in A} \delta(v, r, a, b)$.

This definition is somewhat notationally dense (see [15] for the original concept, framed in terms of metrics). One should interpret this as saying that the Kemeny Rule interprets a vote for ranking v by assigning $\binom{n}{2}$ points to the ranking v, $\binom{n}{2} - 1$ points to any ranking r differing by one switch of places from v (i.e. switching $v(i)$ and $v(i+1)$ for some i), and so on, down to no points to the ranking which reverses v completely. Then one adds up points as usual to determine the 'winning' ranking(s).

[4]Usually one requires the entries to be nonincreasing, but a priori this need not be so.

[5]Variants of this are commonly used in ranking teams in collegiate or high-school sports.

[6]Conitzer introduces these to study procedures which are 'maximum likelihood estimators' when one assumes there is an (unknown) correct ranking for a society, and Zwicker puts them in a more general context (where neutrality may not be a meaningful concept) toward characterizing certain geometrically defined types of rules. Here, however, we study them for their own sake.

KR is particularly important because it is the unique preference function which is neutral and obeys two other important voting theory axioms[7]; see the famous paper [27]. For

Exercise 4. Let $\mathbf{p}$ be a profile for $n = 3$ such that other than $\mathbf{p}(ABC) = 4$ and $\mathbf{p}(BCA) = 3$, $\mathbf{p}(r) = 0$. Show that BC would give B as the winner, but KR gives the winning ranking to be ABC.

On the other hand, every positional scoring rule is a neutral SRSF as well (Proposition 1 of [6]). Given a vote v and a candidate a, let $t(v, a)$ be the number of points that a gets if someone votes v. Then if we denote the ith-place candidate in a ranking r by $r(i)$, we have that

$$s(v,r) = \sum_{i=1}^{m}(m-i)t(v,r(i))$$

turns a positional scoring rule into an SRSF. Intuitively, the SRSF score for a vote v with respect to a ranking r is the sum of points each candidate in ranking r gets in the scoring rule under vote v, weighted by how high the candidate was in ranking r.

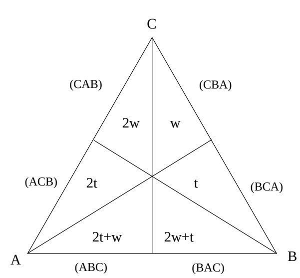
We can make all this quite concrete with three candidates. For a positional scoring rule, if $v = ABC$ and $r = XYZ$, then

$$\begin{aligned} s(v,r) &= \sum_{i=1}^{3}(3-i)t(ABC,r(i)) \\ &= 2t(ABC,X) + t(ABC,Y) + 0 \cdot t(ABC,Z) \\ &= 2t(ABC,X) + t(ABC,Y)\,, \end{aligned}$$

so if we have a system with $\mathbf{w} = (t, w, 0)$, then this yields

r	ABC	ACB	CAB	CBA	BCA	BAC
$s(v,r)$	$2t+w$	$2t$	$2w$	w	t	$2w+t$

We can then visualize it using a triangle[8] where each separating line defines the border between rankings with $X \succ Y$ and vice versa. To use it for other $v' \neq v$, one would permute the whole triangle with the same permutation σ such that $\sigma(v) = v'$. Then computing the SRSF for each $r = XYZ$ can be done visually as well, by taking the dot product of the profile and the weighting triangle with ABC on the XYZ spot.



[7]Consistency, which essentially says that if you join two profiles the outcome should be in the intersection of their outcomes, and being a Condorcet extension, which is a fairly strong way of agreeing with the individual pairwise outcomes which come from $\mathbf{p}$.

[8]This is obviously related to Saari's representation triangle, but here we do not interpret it as a simplex, only as a graphical convenience.

It is instructive to see what plurality looks like under this system for arbitrary n. Since $t(v,r) = 0$ unless $r(i) = v(1)$, in which case we get $s(v,r) = n - i$, the score for r is $\sum_{v \in V} \mathbf{p}(v)s(v,r)$, which is the sum of $n-1$ points for each voter who ranks $r(1)$ first, $n-2$ points for each one who ranks it second, and so forth.

Exercise 5. Try this with several profiles and plurality for $n = 3$, to check that the winning ranking has the (usual) plurality winner as its top-ranked alternative.

2.3. **The Permutahedron and Reversal Symmetry.** The reader will have noticed that input profiles and output scores of SRSFs are both essentially elements of the vector space $\mathbb{Q}S_n$. In fact, *neutral* SRSFs have *all* their $s(v,r)$ basically given by *one* such vector, since $s(\sigma(v), r) = s(v, \sigma^{-1}, r)$. Hence, to analyze neutral SRSFs, we will naturally look at this structure. To be specific, we are equating σ with the ranking r such that $r(\sigma(i)) = X_i$.[9]

It is time to introduce the other major player in our story. A large role has been played in traditional analysis of both the positional scoring rules and the Kemeny rule by yet another symmetry. Intuitively, this is the concept that if everyone reverses *all* their preferences, then the final outcome should essentially be reversed as well.

Let $\rho = (1,n)(2,n-1)\cdots$ be the so-called 'reversal' element of S_n. Then the ranking corresponding to $\rho\sigma$ is the one such that $r(\rho\sigma(i)) = r(n+1-\sigma(i)) = X_i$, or in other words the strict reversal of the ranking corresponding to σ (like $A \succ B \succ C$ is the reversal of $C \succ B \succ A$). For a general ranking v, we denote its reversal by v^ρ; we will use the same notation for the operation of reversing all rankings in a set or changing $\mathbf{p}(v)$ to $\mathbf{p}(v^\rho)$ for all v in a profile.

Definition 6. We say a social preference function f has *reversal symmetry* if $f(\mathbf{p})^\rho = f(\mathbf{p}^\rho)$ for all profiles $\mathbf{p}$.

It is definitely *not* the case that most SRSFs observe this symmetry. For instance, the plurality vote does not do so. Any profile $\mathbf{p}$ with 25% each preferring $A \succ B \succ C$, $A \succ C \succ B$, $C \succ B \succ A$, and $B \succ C \succ A$ will show this. Plurality will cause the two rankings with A first as winners, whether one uses $\mathbf{p}$ or $\mathbf{p}^\rho$, and so reversing the profile to $\mathbf{p}^\rho$ does not reverse the winning rankings.

Examples of rules which do have reversal symmetry are BC and KR. They are symmetric with respect to the following combinatorial object (see for instance [29]).

Definition 7. The n-permutahedron Π_n is the graph with $n!$ vertices, indexed by permutations of the set $\{1,2,\ldots,n\}$ (or elements of S_n, as preferred), and with an edge connecting permutations σ and σ' if and only if $\sigma' = (i,i+1)\sigma$ for some $1 \le i < n$. We call its symmetry group P_n.

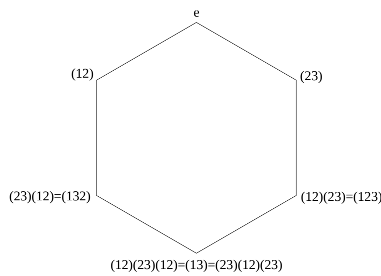
Another way to say this same definition is that the permutahedron is the Cayley graph of the symmetric group S_n for the neighbor-swap generating set

$$\{(1,2),(2,3),\ldots,(n-1,n)\} .$$

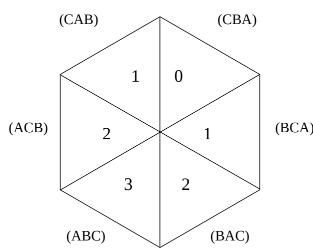
Since ρ simply changes i to $n+1-i$ in a permutation, a neighbor-swap $(i,i+1)$ will become $(n+1-i, n+i)$, so all edges are preserved by ρ, which means this is just the tool for looking at reversal symmetry.

[9]For instance, $\sigma = (1\,2\,3)$ will correspond to $r(1) = r(\sigma(3)) = X_3$, $r(2) = r(\sigma(1)) = X_1$, and $r(3) = r(\sigma(2)) = X_2$, or $X_3 \succ X_1 \succ X_2$.

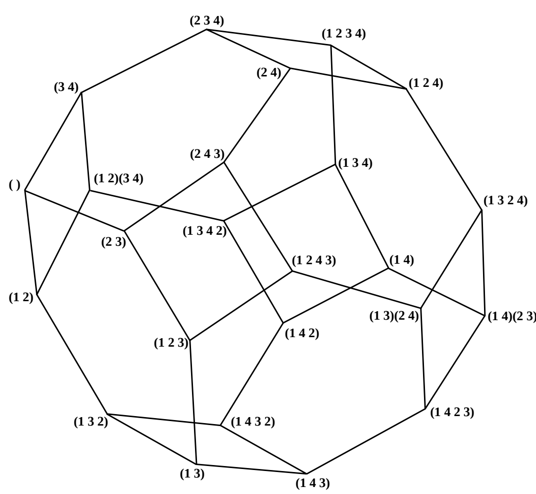
In particular, the 3-permutahedron is in fact the (graph associated to the) regular hexagon, where the vertices are seen as being labeled by permutations of $\{1, 2, 3\}$, written as reduced words:



We can use this to easily visualize neutral SRSFs which satisfy reversal symmetry. Once again, we should think of the score for a ranking r as being a sort of dot product of the profile with the hexagon, except that in this case we see that rankings the same distance from r get the same score. This example gives the Kemeny Rule, for instance.



The permutahedron generalizes naturally.



Since reversal symmetry is important, we may ask what overall symmetry group we have here - that is, what is P_n? Perhaps surprisingly, this seems to exist more in mathematical folklore[10] than otherwise, but it turns out that $P_n \cong S_n \times C_2$ in a very natural way, where C_n is the cyclic group of order n.

Since it has index two, it is clear that $S_n \triangleleft P_n$. In particular, if one thinks of S_n as acting by right multiplication on the permutahedron, this provides a natural inclusion inside P_n. Then, since we already saw that ρ acts on the *left* on the permutahedron, this action defines exactly the subgroup C_2 which gives a direct product, since left and right multiplication will commute!

3. Representation Theory and Voting

We are now ready to discuss how groups will help us. As in [8], we will call the vector space of *all* profiles $M^{(1,1,\dots,1)} = M^{1^n}$, for reasons that will become apparent below; we have already seen that (as vector spaces) $M^{1^n} \cong \mathbb{Q}S_n$. Further, any preference function has its image in M^{1^n} as well. In that case, here is the key observation.

Since f is neutral, we can let $\mathbf{s}$ be the vector of all $s(v, \cdot)$. Then the scores for all rankings r are simply the dot products $\sigma(\mathbf{s}) \cdot \mathbf{p}$. Then we can use the same terminology as [8] and say that $f_\mathbf{s}$ is given by a linear transformation $T_\mathbf{s} : M^{1^n} \to M^{1^n}$.

Furthermore, we have already observed that if we let $\sigma(\mathbf{p}(v)) = \mathbf{p}\left(\sigma^{-1}(v)\right)$, this is compatible with SRSFs (this is a version of a voting concept called *anonymity*, which essentially says that all voters have the same influence). That is, the profile space M^{1^n} is a $\mathbb{Q}S_n$-module. But that is not all! In fact, $T_\mathbf{s}$ is a $\mathbb{Q}S_n$-module homomorphism by neutrality. Finally, by exactly the same argument as in [8], since $\mathbf{p} \in M^{1^n} \cong \mathbb{Q}S_n$, a neutral SRSF is the result of the profile acting on $\mathbf{s}$, so that $T_\mathbf{s}(\mathbf{p}) = \mathbf{ps}$, in the sense of the group rings.

Once we know that $T_\mathbf{s}$ is a $\mathbb{Q}S_n$-module homomorphism, we can use representation theory to find out things about it. Our main tool will be decomposition into irreducible submodules, and the following well-known result:

Schur's Lemma. *If M and N are irreducible G-modules and $g : M \to N$ is a G-module homomorphism, then either $g = 0$ or g is an isomorphism.*

Let's start by analyzing neutral SRSFs when $n = 3$. The irreducible decomposition of $\mathbb{Q}S_3 = M^{(1,1,1)}$ is given by $M^{(1,1,1)} \cong S^{(3)} \oplus 2S^{(2,1)} \oplus S^{(1,1,1)}$, where $S^{(3)}$ and $S^{(1,1,1)}$ are names for special one-dimensional submodules, but each copy of $S^{(2,1)}$ has dimension two. These modules are indexed by the three possible partitions of $n = 3$, a classical result in representation theory of the symmetric group; see the Appendix for more details and the dimensions.

In this case, we can give explicit bases for these, where we use the order of group elements $e, (2\,3), (1\,2\,3), (1\,3), (1\,3\,2), (1\,2)$[11]. Note that the sum of the entries of all the vectors (except the first) are zero; we call such a vector *sum-zero*.

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[10] See [28, 10, 7].

[11] This corresponds to the usual voting theory order of ABC, ACB, CAB, etc.

$S^{(3)}$	$(1,1,1,1,1,1)$
$S_1^{(2,1)}$	$(2,0,-2,-2,0,2)$
	$(0,-2,-2,0,2,2)$
$S_2^{(2,1)}$	$(1,1,-2,1,1,-2)$
	$(-2,1,1,-2,1,1)$
$S^{(1,1,1)}$	$(1,-1,1,-1,1,-1)$

Since $T_{\mathbf{s}}$ is a G-module homomorphism, Schur's Lemma tells us that $S^{(3)}$ and $S^{(1,1,1)}$ go to themselves, in both cases obviously by multiplication by some scalar, while each $S^{(2,1)}$ can go to any linear combination of the two $S^{(2,1)}$ components[12]. This immediately means that a general SRSF has six degrees of freedom, as is to be expected since $\mathbf{s}$ is a six-dimensional vector.

However, we can say more. For instance, it is entirely reasonable to ignore the piece of $T_{\mathbf{s}}$ coming from $S^{(3)}$, since it will only add the same amount to the score for each ranking r, so we will consider all such $\mathbf{s}$ to be equivalent[13]. What kind of $\mathbf{s}$ will give this? Clearly one which is itself sum-zero.

Similarly, since we only care about the *relative* scores, and not the actual scores, we can always take away one additional dimension when discussing how many procedures there are (think of this as taking away a diagonal map). More precisely, the relative scores (and hence ranking of the rankings) for a given profile under $T_{\mathbf{s}}$ and $T_{\mathbf{s}'}$ with $\mathbf{s}$ and $\mathbf{s}'$ sum-zero will be identical if one is a (positive) multiple of the other, which we will indicate by $s \sim s'$[14]. We will say that two neutral SRSFs are *essentially different* if they are *not* equivalent in this way.

Proposition 8. *The space of essentially different neutral SRSFs for $n = 3$ is four dimensional, and is given by $\mathbf{s}$ which are sum-zero (equivalent up to homogeneity).*

In this case, plurality has $\mathbf{s} \sim (2,2,-1,-1,-1,-1)$; even though it looks quite different to the voter, we will consider the 'reversed' plurality $\mathbf{s}' \sim (-2,-2,1,1,1,1)$ to be equivalent, since it will literally have a reversed outcome which can be derived from plurality. (We are intentionally ignoring issues such as unanimity to get the broadest possible result right now.)

Now we bring in the permutahedron. Since $S_n \lhd P_n$, all of these modules are naturally P_3-modules as well, by the action of ρ on $\mathbf{p}$. It is not hard to see that $\left(S_1^{(2,1)}\right)^\rho = -S_1^{(2,1)}$ while $\left(S_2^{(2,1)}\right)^\rho = S_2^{(2,1)}$, so these isomorphic S_3-modules are *not* equivalent as P_3-modules!

The implication is that with reversal symmetry, each $S^{(2,1)}$ piece must *go to itself*; even for $n = 3$, this makes a big difference, since then each component can only go to a scalar multiple of itself.

Proposition 9. *The space of essentially different neutral SRSFs for $n = 3$ which obey reversal symmetry is two dimensional.*

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[12]Note, however, the extremely important point that the first basis vector of each must go to a linear combination of the first basis vectors. This is because these vectors have symmetry under any σ which switches alternatives B and C, while the others do not, and these are S_3-module homomorphisms.

[13]The argument is identical to that in Section 6 of [8].

[14]Again, as in [8] and elsewhere. Note that we are not directly interested in the $\mathbf{s}$, so we do not reprove analogues of the theorems in Section 6 of that paper.

This space is given by all equivalence classes of $\mathbf{s}$ for which $\mathbf{s}$ is sum-zero and $s(\rho\sigma\rho, r) = s(\sigma, r)$, which we may think of naively as saying $s(ACB, ABC) = s(BAC, ABC)$ and $s(CAB, ABC) = s(BCA, ABC)$. The point is that these spaces are *computable*, not just theoretical.

Alternately, to go further in classifying them, one could characterize all (not just essentially different) neutral SRSFs which obey reversal symmetry as being given by linear combinations of the identity maps between each of the four irreducible components (and then use algebra to discover more). We give an example of this to lead to our main result.

One candidate for being a complete tie is any profile in $S_2^{(2,1)}$; after all, these profiles have complete ties for any head-to-head matchup between alternatives. So we could look at the subspace of procedures for which this map is the zero map.

Proposition 10. *The space of essentially different neutral SRSFs for $n = 3$ which obey reversal symmetry and ignore complete head-to-head ties is a one-dimensional family of procedures.*

For $n = 3$, one may think of this as saying that, in the hexagon, $b = 1 = -c$ and $a = -d$. The next exercise confirms the main theorem of this paper - that this space includes both the Borda Count and Kemeny Rule.

Exercise 11. The KR is given by $a = 3$ and BC is given by $a = 2$.

Note that the condition $a \geq 1$ is equivalent to a unanimity condition; that is, if $a \geq 1$, then any profile in which all voters have the same preference will have that preference as a winning ranking under $f_{\mathbf{s}}$. Even here, and for $n = 3$, though, it is possible for SRSFs in this space to give actual outcomes (winning preference orders) which are different from both BC and KR.

Example 12. With $\mathbf{p} = (1, 2, 5, 0, 0, 0)$, the Kemeny Rule and Borda Count both give CAB as the winning outcome, but with $a = 1.5$ the result is ACB.

However, it turns out that if $2 < a < 3$, this is not possible − all SRSFs of this type will have the same outcome as KR or BC (or both). Demonstrating this is a standard chase of inequalities to yield contradictions, so we omit the proof. The point is that the algebraic access to these types of procedures makes it possible to do further analysis in various directions of interest to voting theorists.

4. General Statements

The rest of the paper is concerned with stating and proving similar statements for all n. The main results may be summarized as follows.

- The space of essentially different neutral SRSFs which ignore all complete head-to-head ties is $\frac{1}{2}(n+1)(n-2) = \frac{1}{2}(n^2 - n + 2)$ dimensional (Theorem 13).
- If these also have reversal symmetry, we are reduced to $\frac{1}{4}(n^2-5)$ or $\frac{1}{4}(n^2-4)$ dimensions for odd and even n, respectively, which is about half as many (Theorem 17).
- Unsurprisingly, there are n dimensions of positional scoring rules, $\left\lfloor \frac{n}{2} \right\rfloor$ of which obey reversal symmetry (Theorems 18 and 21).
- An SRSF which is a scoring rule *and* ignores complete head-to-head ties is essentially the same as the Borda Count or its reversal (Corollary 22).

- Suppose an SRSF ignores all complete head-to-head ties *and* does not have any complete head-to-head tie component in its vector of scores (for any profile). Then this procedure lies on a one-dimensional continuum with the Borda Count and Kemeny Rule (Theorem 23).

We will first discuss needed parts of the decomposition of M^{1^n} for general n, and then state the theorems precisely.

4.1. **Voting-Theoretic Decompositions for General n.** We have already noted that the key to understanding the SRSF space is by decomposing the profile space $M^{1^n} \cong \mathbb{Q}S_n$ as S_n- and P_n-modules. We note (referring to the Appendix for notation) that the irreducible decomposition of M^{1^n} under S_n has, among other pieces, $n-1$ copies of $S^{(n-1,1)}$ and $\binom{n-1}{2}$ copies of $S^{(n-2,1,1)}$. As we will see, these are the only ones we will need to deal with.

There are various ways to think of $(n-1)S^{(n-1,1)}$, but the classification in [18, 19] as the Borda (B), Alternating (Alt), and Symmetric (Sym) components is most useful. As with our $n=3$ example, these are profile *differentials* (i.e. they are sum-zero), since they are orthogonal to $S^{(n)}$. We summarize them in the following table, with unmentioned values being zero. In all cases, these have the structure that the sum of each of these components for all candidates is zero, so that each of them is $n-1$-dimensional.

B_X	$B_X(r) = n+1-2k$	if $r(X) = k$
$Alt_{j,X}$	$Alt_{j,X}(r) = n-1$	if $r(X) = j$
$2 \leq j \leq \frac{n-1}{2}$	$Alt_{j,X}(r) = 1-n$	if $r(X) = n+1-j$
	$Alt_{j,X}(r) = 2j-n-1$	if $r(X) = 1$
	$Alt_{j,X}(r) = 1+n-2j$	if $r(X) = n$
$Sym_{j,X}$	$Sym_{j,X}(r) = 1$	if $r(X) = j$ or $n+1-j$
$2 \leq j < \frac{n+1}{2}$	$Sym_{j,X}(r) = -1$	if $r(X) = 1$ or n
$Sym_{\frac{n+1}{2},X}$	$Sym_{\frac{n+1}{2},X}(r) = 2$	if $r(X) = \frac{n+1}{2}$
	$Sym_{\frac{n+1}{2},X}(r) = -1$	if $r(X) = 1$ or n

These components are also irreducible P_n-modules, which makes them so useful. It is not hard to see that under reversal symmetry the B and Alt components reverse sign, while the Sym components are unchanged, so us the notation $S^{(n-1,1),+}$ for the isomorphism class of the Sym modules, while the B and Alt components are called $S^{(n-1,1),-}$.

Most of the $S^{(n-2,1,1)}$ component will vanish for most of the procedures we discuss, so we will only identify the ones corresponding to $(1,-1,1,-1,1,-1)$ in the $n=3$ example. For each pair $\{X,Y\}$ of candidates, we will define C_{XY}, the Condorcet component[15] The name comes from the fact that the pairwise votes in $(1,-1,1,-1,1,-1)$ end up giving $A \succ B$, $B \succ C$, $C \succ A$, an example which the Marquis de Condorcet used to great advantage in promoting his ideas of voting.

More precisely, let $\{XY1\}$ denote the set of all rankings which begin $X \succ Y$, let $\{XY2\}$ denote the set of all rankings which begin with $X \succ ? \succ Y$, and so on up

[15]This is obviously indebted to Saari's original Condorcet component and Zwicker's 'spin' component ([30]) as well as Saari's C_{XY} in Section 6 of [18], and indeed the space only is different from the latter when $n \geq 5$, which is probably why they were not noticed until recently. See also Sections 4.4.3 and 4.5 of [20] where they are at least implicit, though the 'old' Condorcet components still take pride of place.

through $\{XY(n-1)\}$. Then for all rotations of the elements in $\{XYi\}$ (such as ABC, BCA, CAB for three candidates), we assign $n-2i$ voters to those rankings, and this is C_{XY}.

Notice that $\{XYi\}$ is simply the reversal of all $\{XY(n-i)\}$, so there is redundancy. Notice also that for $n=3$, this clearly gives the usual Condorcet component, while for $n=4$ it gives the Condorcet component in the form of the C_{XY}s of Saari, as for $i=2$ we get zero. A convenient basis of dimension $\binom{n-1}{2}$ is given by

$$C_{A_1A_2}, C_{A_1A_3}, \cdots, C_{A_1A_{n-1}}, C_{A_2A_3}, \cdots, C_{A_2A_{n-1}}, \cdots, C_{A_{n-2}A_{n-1}}$$

where one notes that holding X *or* Y constant and summing over all candidates in the other variable gives zero.

We require two final concepts, both from [8].

We define the *pairs map* $P : M^{1^n} \to M^{1^n}$ to be the linear transformation that sends a basis vector of M^{1^n} to the sum of all such vectors whose top two candidates are in the same order as in the input vector. For instance, if $\mathbf{p}(BACD) = 1$ and $\mathbf{p}(XYZW) = 0$ for all other rankings, $P(\mathbf{p})(XYZW) = 1$ if XY is one of the (ordered) pairs BA, BC, BD, AC, AD, CD and $P(\mathbf{p})(XYZW) = 0$ otherwise. This naturally encodes all the usual information we associate with comparing candidates on a pairwise basis – for instance, in the Borda Count and Kemeny Rule[16].

The *effective space* of a linear transformation T is the orthogonal complement to the kernel of T – which determines what will *not* be in the kernel. In essence, this is the subspace of elements of the domain of T (and in our context, f) which have no part simply sent to zero (in our context, a complete tie). Since we can often compute the dimension of this space, it will help us compute the dimensions of the sets of procedures. In particular (see the discussion before Theorem 6 of [8]), the effective space of P is isomorphic as an S_n-module to $S^{(n)} \oplus S^{(n-1,1)} \oplus S^{(n-2,1,1)}$. Momentarily we will see this space is comprised precisely of the components we already know about.

What is important is that any procedure which relies only on pairwise information will automatically send all complete head-to-head tie portions of the profile to zero, which means its kernel contains the kernel of P, and hence its effective space will be (possibly a subspace of) the effective space of P. We are now ready to exploit all of this.

4.2. **Statements of Theorems.** Most of the proofs will be postponed, but particularly instructive examples and proofs will be interspersed.

Theorem 13. *The effective space of any neutral SRSF which relies only on pairwise information is $S^{(n)} \oplus B \oplus C$, where the latter two are the Borda and Condorcet spaces defined above. Its image might lie in any piece of $S^{(n)} \oplus (n-1)S^{(n-1,1)} \oplus \binom{n-1}{2}S^{(n-2,1,1)}$, however.*

This can be proved most easily by noting that the KR and BC both kill complete head-to-head ties, but take B and C to multiples of themselves (hence B and C must be the specific copies of $S^{(n-1,1)}$ and $S^{(n-2,1,1)}$ in question).

Proposition 14. *The BC sends any Borda component B_X to a multiple of B_X.*

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[16]But also others familiar to voting theorists, like the Condorcet, Simpson, and Copeland schemes.

Proposition 15. *The KR sends any Condorcet component C_{XY} to a multiple of C_{XY}.*

Playing with this can lead to amusing examples.

Example 16. If $\mathbf{s} = (0, -1, -1, 0, 1, 1)$, show that $f_{\mathbf{s}}$ sends B_A to $-2R_A$, but is a neutral SRSF relying only on pairwise information.

A possible interpretation of this is that a voter profile which overwhelmingly approves of rankings with A ahead of other rankings would have an outcome overwhelmingly favoring any ranking with A in second place ahead of other rankings!

Theorem 17. *With reversal symmetry obeyed, however, the image is reduced to being in $S^{(n)} \oplus \frac{1}{2}(n-1)S^{(n-1,1),-} \oplus \binom{n-1}{2}S^{(n-2,1,1),-}$.*

Recall from Section 4.1 that the irreducible isomorphism classes $S^{(n-1,1),\pm}$ and $S^{(n-2,1,1),\pm}$ are P_n-modules, essentially differing in the same way that $S^{(2,1)}_{1,2}$ differed in Proposition 10.The proof is basically by looking at Theorem 25 and noting that $B \cong S^{(n-1,1),-}$ and $C \cong S^{(n-2,1,1),-}$ in that notation.

What about positional scoring rules? We state the SRSF analogue of the remark before Theorem 4 of [8] (the proof is essentially the same).

Theorem 18. *The effective space of any SRSF which is a positional scoring rule (unless its vector of weights is sum-zero) is isomorphic to $S^{(n)} \oplus S^{(n-1,1)}$, where the $S^{(n-1,1)}$ component may be any piece of the whole $(n-1)S^{(n-1,1)}$ piece of $\mathbb{Q}S_n$.*

In fact, the copy of $S^{(n-1,1)}$ will depend on the structure of $\mathbf{s}$, as pointed out there. Nonetheless, the SRSF point of view is quite enlightening, as we will see in a moment.

Proposition 19. *Any profile orthogonal to $S^{(n)} \oplus (n-1)S^{(n-1,1)}$ has the property that it is sum-zero not just as a whole, but also for the subset of rankings r such that $r(j) = X$, for any j and any X.*

Proof. This follows from the structure of the $S^{(n-1,1)}$ components, which has a basis of vectors $\mathbf{p}$ such that $\mathbf{p}(v)$ is the same for all v with $v(j) = X$ (we saw this in Section 4.1). Combining these with the $S^{(n)}$ basis, and one sees that the vector which has value 1 for all $v(j) = X$ and zeros elsewhere is in this span; being orthogonal to this is the content of the proposition. □

As a result, the score allocated to ranking r from any profile orthogonal to this will be

$$\sum_{k=1}^{n} \left(\sum_{i=1}^{n-1} \left((n-i) \sum_{v(k)=X} t(v, r(i)) \right) \right)$$

where the innermost sum is counted with (possibly negative) multiplicity, and hence must be zero by the proposition.

Proposition 20. *The image of any positional scoring rule will be in $S^{(n)} \oplus B$.*

Just as moving to the algebraic viewpoint gives us cardinal, not just ordinal information, this gives us more information than before. Namely, positional scoring rules are extremely limited in their outcome potential; depending on your point of view, this might be good or bad. It certainly limits the types of ties one can have, for instance.

Theorem 21. *With reversal symmetry obeyed, however, a positional scoring rule must have its effective space be* $S^{(n)} \oplus S^{(n-1,1),-}$.

This certainly makes sense, as one would expect the weights $x_i + x_{n+1-i}$ to be invariant with respect to i in that case, or to equal zero if $S^{(n)} \to 0$. And it makes the following well-known fact even simpler, since the intersection of $(n-1)S^{(n-1,1)}$ and $B \oplus C$ is B.

Corollary 22. *An SRSF which is both a scoring rule* and *relies only on pairwise information has an effective space and image of* $S^{(n)} \oplus B$. *This must be essentially the same as the Borda count (remember, this includes the reversal of BC).*

4.3. **The Borda Count and the Kemeny Rule.** Here we finally state and prove the main theorem, as well as give some context. Before we go on, though, we step back a bit to give some motivation.

Even if a neutral SRSF has lots of nice properties, there are still weird things that can happen. For instance, with one obeying reversal symmetry, Borda B_X profiles could be sent to things containing pieces like the Alternating profiles; for an example with $n = 4$, one could send B_X to Alt_X, which has -1 voters for each ranking with X first, $+1$ voters for each with X last, $+3$ for each ranking with X second, and -3 for each ranking with X third. Given that B_X expresses very strong support for X, it's not clear that it's a great move to have that now interpreted as support for X in second place.

In fact, we can even have a procedure for $n = 4$ which kills off the Condorcet component, but sends B_A to $-40Alt_A$, going from overwhelming approval for A over all others to overwhelming approval for any profile with A in third place, some for ones with A in first, but the least for those with A in second!

Of course, this would have $s(ABCD, ABCD) = 0$, $s(BACD, ABCD) = -2$ but $s(ACBD, ABCD) = 3$, and $s(BDAC, ABCD) = -5$, so it is not a procedure one would actually use – but that is not the point. *Any* neutral, reversal-symmetric SRSF $f_{\mathbf{s}'}$ such that $\mathbf{s}'$ had a component of this $\mathbf{s}$ in it would incorporate some of that strange behavior, and we probably want to avoid that in normal interpretations of choice procedures. This is the essence of the algebraic/geometric point of view of voting theory.

Hence, once we have bothered to *get* a reasonable effective space of profiles with clear intent in their structure by relying only on pairwise information, we will probably want to send that effective space to itself.

Theorem 23. *Suppose a neutral SRSF relies only on pairwise information and in addition has an outcome which does not have any complete head-to-head tie component. Then (up to essential difference) this SRSF is on a one-dimensional continuum with the Borda Count and Kemeny Rule. If in addition the Condorcet component goes to zero, the rule is the Borda Count.*

By applying the additional hypothesis to Theorem 13 (which kills $S^{(n)}$), we see that such an SRSF must go from $B \oplus C$ to itself, hence the space is one-dimensional up to essential difference. To prove that the BC and KR are in fact on this continuum, simply collate Propositions 14 and 15, Corollary 22, and the following result.

Proposition 24. *The KR sends any Borda component* B_X *to a multiple of* B_X.

We have already given explicit shape to this continuum for $n = 3$, but we will normalize it differently here for comparison with the case for $n = 4$. For convenience, we take $s(r, r) = 1$ and $s(r, \rho(r)) = -1$. Then for $n = 3$ the continuum with parameter t goes from $[1, t, -t, -1]$ with $t = 1/3$ being Kemeny and $t = 1/2$ being Borda.

For $n = 4$ the continuum is more subtle. Assuming that $s(r, r) = 1$ and $s(r, \rho(r)) = -1$ as before, for a ranking r which is one neighbor swap away from v (as in the comment after Definition 3), we would have $s(v, r) = 2t$. For most r at distance two we would have t points, but $s(XYZW, YXWZ) = 4t - 1$. We would have $s(XYZW, YWXZ) = s(XYZW, ZXWY) = 0$, but the others at distance three having $\pm(3s - 1)$, and those further away having negative points. The Kemeny Rule is at $t = 1/3$, the Borda Count at $t = 2/5$.

Notice that this spectrum already has more complexity; for instance, $4t - 1$ could be greater than or less than t depending on whether t was greater than $1/3$ or not. We do get some additional constraints with a weak form of Pareto (that the partial order on the permutahedron is respected by $\mathbf{s}$); in this case, we must have $1/4 \leq t \leq 1/2$.

What is particularly interesting about this is that there are reasonable (from this point of view) methods both 'between' KR and BC, but also on either side of them along the continuum. Why might one be interested in such methods?

Historically, those who study voting methods (or social welfare functions, or social preference functions) from this linear algebraic/geometric perspective have found that the intense symmetry associated with the Borda Count made it a prime candidate for real-life use. For instance, [8] was motivated largely by trying to analyze useful analogues to the BC for partial rankings. And indeed, the BC has the very salutary effect of taking the component of a profile which includes Condorcet cycles (such as $A \succ B \succ C \succ A$ mentioned earlier) and treating it as a complete tie[17]. For methods intended to provide a winner or set of winners, this seems like the most reasonable thing to do, even if it means that a candidate who beats all other candidates head-to-head might lose the election[18].

However, choice theory is about *choice*, not just winners. In particular, one can imagine many situations where a voter might really be invested in the *entire* ranking - for instance, in an election for an advisory board, with a clear succession needed in the event a board member cannot attend a meeting or is incapacitated. In such a situation, it is entirely reasonable for voters for $ABCD$ to say that they would prefer $BCAD$, where at least most of the succession is preserved even if their favorite is next-to-last, to $ADCB$ where their first-choice candidate wins but the rest of it is counter to their preference. In fact, it might be appropriate to leave a component of a profile which looks like $(1, 0, 1, 0, 1, 0)$ as a tie between the *rankings* ABC, BCA, CAB rather than a tie between *candidates* A, B, C; this would potentially be slightly preferable to the rankings ACB, CBA, BAC for such voters.

The Borda Count is the dividing line between Condorcet components being sent to themselves or their negatives, so this might cause one to reject social preference functions 'beyond' it (for $n = 4$, with $t < 2/5$); similarly, perhaps for procedures

[17]That is, the Borda Count sends $C \to 0$, as we pointed out earlier.

[18]This is called the *Condorcet criterion*, and is the locus of much of the disagreement about methods to use in social choice theory.

'beyond' KR one might say that too much influence is given to something which is like a tie. However one might judge, the permutahedron has given the tools to start making those choices.

Another example of where this might be useful is in considering manipulation. It is 'classical' (originally due independently to Gibbard and Satterthwaite; see [25] for a comprehensive survey) that situations exist in nearly any choice system where a voter can cast a vote other than his or her actual preference and come out with a more preferred result. Geometric-algebraic methods have done much work analyzing this for BC and KR (see for instance [21]); one might expect this batch of procedures to have similarly preferable behavior with respect to manipulation.

There is one additional possible interpretation of this worth considering. The permutahedron is an abstract combinatorial object, but it may be embedded consistently in space in many ways. Zwicker has pointed out (in [32]; see also [23]) that one of the equivalent weighting vectors **s** for both KR and BC come from square distances between its vertices in different embeddings. Might there be a way to think of some of the other methods along this spectrum as part of the continuum stretching (for $n = 3$) the regular hexagon to the permutahedral vertices of the cube? (And if so, can we find a geometric interpretation of $t > 1/2$?) Ideally, this would give a natural connection to the representation theory as well.

5. Future Work and Acknowledgments

It is clear there are many opportunities for further work here. For instance:

- What about things like Pareto and unanimity? We only touched on this.
- We have seen that the continuum of procedures can be different from BC and KR. How different? To what extent do they share the desirable properties of each?
- What about truncated, tied, or incomplete preferences in this context?
- What about voting on placement in an order where the beginning doesn't matter (around a table, for instance)? Or one where it doesn't matter, except for one special spot? Or one where the order doesn't matter, but orientation does?
- What about voting with respect to the symmetries of some arbitrary graph on a set of alternatives?
- Can one give explicit form to the geometric models toward the end of Section 4.3?

We want to explicitly point out that without the use of mathematical software (we used Sage, [24]), discovering these rather subtle patterns would have been much harder, particularly when it came to more than $n = 4$ candidates. As Archimedes pointed out[19] it is much easier to prove something once you know what to prove!

There are many people to thank, but we point out several in particular.

- Mike Jones of Math Reviews for giving the opportunity to present a very early version of this work.
- Bill Zwicker for pointing out the connection to Kemeny and the Borda Count, and for many valuable references and discussion.

[19] With respect to both Democritus and Eudoxus deserving credit for showing that a cone or pyramid has one-third the volume of the respective cylinder, see e.g. [12] for discussion.

- Mike Orrison for enthusiastic support and references for the representation theory.
- The Gordon College Faculty Development Committee for the sabbatical leave during which much of this research took shape.

6. Appendix

The appendix has two sections. In the first, we review some representation theory and perform some needed computations. In the second, we provide several proofs of properties of some preference functions which would otherwise have interrupted the exposition.

6.1. Representations of the Permutahedron.

Our main goal is Theorem 25, but we will also collate several other results about the representation theory of P_n and S_n, providing proofs where these are not well-known. The book [14] is a canonical reference, but [9] is in a more closely related context and we echo its notation.

It is classical that the irreducible representations of S_n (over a field of characteristic zero) are classified by partitions λ of n, using e.g. Young tableaux, each called S^λ. For instance, for $n = 4$ there are precisely five, labeled $S^{(1,1,1,1)}$, $S^{(2,1,1)}$, $S^{(2,2)}$, $S^{(3,1)}$, and $S^{(4)}$. It is also classical that the regular representation $\mathbb{Q}S_n$ decomposes, as a S_n-module, as $\bigoplus_\lambda \dim(S^\lambda)S^\lambda$.

The regular representation of S_n is only one instance of a more general object. Namely, it is the representation given by the action of S_n on the vector space $\mathbb{Q}X$, where X in this case happens to be the set of permutations of $\{1, 2, \ldots, n\}$. This set X may also be considered as the set of vertices of the permutahedron. In that case there is also a $P_n \cong S_n \times C_2$ action on X, and hence $\mathbb{Q}X$ has a P_n-module structure as well.

Since P_n has such a nice structure, we know (see e.g. [1], Example 15.2) that each S^λ will be isomorphic (as an S_n-module) to two irreducible P_n-modules, which we will call $S^{\lambda,+}$ and $S^{\lambda,-}$ to indicate how ρ acts on them (namely, $\rho S^{\lambda,+} = S^{\lambda,+}$ but $\rho S^{\lambda,-} = -S^{\lambda,-}$).

So our goal is to understand enough of the decomposition of $\mathbb{Q}X$ to get information about voting. A first piece of information is that the number of copies of $S^{\lambda,+}$ and $S^{\lambda,-}$ will be the same as the dimension of S^λ given by the hook-length formula. Unfortunately, trying to use this directly for the whole decomposition is challenging.

But we don't really care about most of the decomposition, as voting-theoretically it lies in some kernel (though it might be useful to look at those pieces for other reasons). What we really care about are the $S^{(n-1,1)}$ and $S^{(n-2,1,1)}$ components, which are the ones that pairwise-respecting procedures and points-based procedures are affected by.

Theorem 25. *For $n > 3$, the decomposition of $\mathbb{Q}X$ as a P_n-module includes exactly the following number of copies of these irreducible submodules:*

Irreducible	*Number*
$S^{(n-1,1),+}$	$\frac{1}{2}\left(n-1-\left(\frac{1+(-1)^n}{2}\right)\right)$
$S^{(n-1,1),-}$	$\frac{1}{2}\left(n-1+\left(\frac{1+(-1)^n}{2}\right)\right)$
$S^{(n-2,1,1),+}$	$\frac{1}{2}\left(\binom{n-1}{2}-\lfloor\frac{n-1}{2}\rfloor\right)$
$S^{(n-2,1,1),-}$	$\frac{1}{2}\left(\binom{n-1}{2}+\lfloor\frac{n-1}{2}\rfloor\right)$

This result is all we need to use in our theorems. We give these numbers for small n in the following table.

n	$\#S^{(n-1,1),+}$	$\#S^{(n-1,1),-}$	$\#S^{(n-2,1,1),+}$	$\#S^{(n-2,1,1),-}$
3	1	1	0	1
4	1	2	1	2
5	2	2	2	4
6	2	3	4	6
7	3	3	6	9
8	3	4	9	12
9	4	4	12	16

Hence the decomposition of $M^{(n-1,1)}$ is by $\frac{n-1}{2}$ Symmetric components and $\frac{n-1}{2}$ Borda/Alternating components when n is odd, and $\frac{n}{2}-1$ and $\frac{n}{2}$ when n is even. And the decomposition of $M^{(n-1,1,1)}$ is as separating $\binom{n-1}{2}$ into pieces which are similarly distant.

The following steps will prove the theorem. Let π be the character of the P_n-module $\mathbb{Q}X$.

- Compute π.
- Compute the inner product (π,χ) on the space of class functions for a general character χ of an irreducible P_n-module.
- Compute (π,χ) for the specific χ we care about for the theorem.
- Apply those computations to the size of $S^{(n-1,1)}$ and $S^{(n-2,1,1)}$ to get the theorem.

For a given $g\in P_n$, the value $\pi(g)$ of the character π of $\mathbb{Q}X$ is the number of fixed points of X under that element (conjugacy class) of P_n ([1], Example 15.4). We write a generic element g as either $g=(\sigma,e)$ or $g=(\sigma,\rho)$, where $\sigma\in S_n$ and ρ is the reversing element mentioned earlier.

It should be clear that $\pi(e,e)=n!$, since all vertices are fixed under the identity. It is not too much harder to see that $\pi(\sigma,e)=0$ if $\sigma\neq e$, since the action of a group on itself is transitive.

For the action of the other elements of P_n, we will look more closely at what is going on. Pick an arbitrary vertex p of the permutahedron; for the purposes of the action (left or right), this should be thought of as a permutation of the set $\{1,2,\ldots,n\}$. For p to be a fixed point for $g=(\sigma,\rho)$, it must be the case that (as permutations) $p=\rho p\sigma$. That is, for each $1\le i\le n$, we must have that $p(i)=n+1-p(\sigma(i))$, or $p(\sigma(i))=n+1-p(i)$. But then $p(\sigma(i))=n+1-p(\sigma(\sigma(i)))$ as well, so that $p(i)=p(\sigma(\sigma(i)))$, which by transitivity means $\sigma(\sigma(i))=i$ for all i, which means σ has order two.

This narrows σ down to permutations made up of disjoint transposes (j,k). Further, since $p(i)+p(\sigma(i))=n+1$, if $\sigma(i)=i$ for some i, then $p(i)=\frac{n+1}{2}$, and there can be only one such i. Hence σ is a permutation made up of as many disjoint transposes as possible, which is the cycle decomposition type of ρ; since the cycle decomposition type determines the conjugacy class of a permutation, σ must be in the conjugacy class of ρ! Otherwise there are no fixed points at all.

To simplify the computation if there are, assume $\sigma = \rho$. Then any p which has $p(i) + p(n+1-i) = n+1$ for all i will work. Once we have chosen $p(i)$ for $1 \le i \le \lfloor \frac{n}{2} \rfloor$, that fixes the others. We can choose $p(1)$ to be anything except $\frac{n+1}{2}$ (if that is an integer), which is $2\lfloor \frac{n}{2} \rfloor$ choices, and which then removes $p(n)$ from consideration; then $p(2)$ can be any of the remaining $2\left(\lfloor \frac{n}{2} \rfloor - 1\right)$ choices, and so on. Thus the number of fixed points for $g = (\rho, \rho)$ is $2^{\lfloor \frac{n}{2} \rfloor} \lfloor \frac{n}{2} \rfloor !$

To summarize, $\pi(e,e) = n!$, $\pi(\rho,\rho) = \left(\lfloor \frac{n}{2} \rfloor !\right) 2^{\lfloor \frac{n}{2} \rfloor}$, and $\pi(g) = 0$ for all other elements of the group. Let $f(g)$ be the size of the conjugacy class of g. The conjugacy class of the identity is always just itself, while the conjugacy class of (ρ,ρ) is the set of all (σ,ρ) where σ have the same cycle type as ρ. The easiest way to think of such a σ is as a permutation which then has parentheses put every two entries, yielding $\lfloor \frac{n}{2} \rfloor$ pairs; then we need to divide by the number of symmetries of this, of which there are 2 for each pair, and then divide by the permutations of the pairs.

Now decomposing the character π with respect to any irreducible character χ can be done directly:

$$(\pi,\chi) = \frac{1}{2\cdot n!}\left(\pi(e,e)\cdot f(e,e)\cdot \chi(e,e) + \pi(\rho,\rho)\cdot f(\rho,\rho)\cdot \chi(\rho,\rho) + 0\right) =$$

$$\frac{1}{2\cdot n!}\left(n!\cdot \chi(e,e) + \left(\left\lfloor \frac{n}{2} \right\rfloor !\right) 2^{\lfloor \frac{n}{2} \rfloor} \cdot \frac{n!}{\left(\lfloor \frac{n}{2} \rfloor !\right) 2^{\lfloor \frac{n}{2} \rfloor}} \chi(\rho,\rho)\right) = \frac{1}{2}\left(\chi(e,e) + \chi(\rho,\rho)\right)$$

The following two propositions turn out to be enough to say what happens to everything we care about with respect to positional scoring rules and pairs-respecting rules.

Proposition 26. *If $\chi = \chi_{S^{(n-1,1),-}}$, then $\chi(\rho,\rho) = \left(\frac{1+(-1)^n}{2}\right)$, which is to say it alternates between 0 and 1 for n odd and even.*

Proposition 27. *If $\chi = \chi_{S^{(n-2,1,1),-}}$, then $\chi(\rho,\rho) = \lfloor \frac{n-1}{2} \rfloor$, which is to say it goes through positive integers in order and repeats each value twice, once for n odd and once for n even.*

Before proving these statements, we briefly finish the proof of Theorem 25. We already know that $\chi_{S^{(n-1,1),\pm}}(e,e) = n-1$ and $\chi_{S^{(n-2,1,1),\pm}}(e,e) = \binom{n-1}{2}$. For the $+$ components, the theorem is immediate. For same calculation with the $-$ components, it suffices to recall that $\chi_{S^{\lambda,-}}(\sigma,\rho) = -\chi_{S^{\lambda,+}}(\sigma,\rho)$.

Proof of Proposition 26. We look at the Borda component as being a typical example of $S^{(n-1,1),-}$. We know that $\chi(\rho,\rho)$ is the trace of the matrix given by the action of ρ on the right *and* the left of the permutahedron. We use the usual basis of $B_{A_1}, \cdots, B_{A_{n-1}}$.

Conjugation by ρ is the 'swap' automorphism. It turns out that this sends a ranking with A_j in the ith position to one with A_{n+1-j} in the $n+1-i$th position, as we noted when calculating fixed points, where $q(i) = n+1-p(n+1-i)$. But this automatically means that B_{A_i} is sent under this action to $-B_{A_{n+1-i}}$. Combining this with the fact that $-B_{A_n} = \sum_{i\neq n} B_{A_1}$, that means the matrix looks like

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & -1 \\ 1 & 0 & 0 & \dots & -1 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & -1 & \dots & 0 & 0 \\ 1 & -1 & 0 & \dots & 0 & 0 \end{pmatrix}$$

which will clearly have the correct trace. □

Proof of Proposition 27. We look here at the (new) C_{XY} as a typical example of $S^{(n-2,1,1),-}$. We once again will look at swapping as the action, and will use the basis mentioned above, $C_{A_1A_2}, C_{A_1A_3}, \cdots, C_{A_1A_{n-1}}, C_{A_2A_3}, \cdots, C_{A_2A_{n-1}}, \cdots, C_{A_{n-2}A_{n-1}}$. Using the same argument as above, we see that if $A_i \succ A_j$ originally, then after swapping it is the case that $A_{n+1-j} \succ A_{n+1-i}$, so that $C_{A_iA_j} \to C_{A_{n+1-j}A_{n+1-i}}$. Now, since our basis consists of ones where $i < j$, that means that $n+1-j < n+1-i$ as well. These are all also elements of the basis, except if $n+1-i = n$, in which case $i = 1$. The only way this will contribute to the trace is if $i = n+1-j$ and $j = n+1-i$ (or $i+j = n+1$), or possibly if $i = 1$.

When $i+j = n+1$, this contributes one to the trace. But for $0 < i < j < n$, the only pairs are for $i+j = n+1$ with $i > 1$, which means we only have to count these. So for odd n we get get one pair for each integer $2 \leq i < \frac{n}{2}$, which leaves $\left\lfloor \frac{n}{2} \right\rfloor - 1$. When $i = 1$, we need to get $C_{A_{n+1-j}A_n}$ in terms of the basis. But this is the same as

$$-\sum_{k\neq n+1-j,n} C_{A_{n+1-j}A_k} = -\sum_{n+1-j<k<n} C_{A_{n+1-j}A_k} + \sum_{0<k<n+1-j} C_{A_kA_{n+1-j}}\,.$$

This contributes to the trace precisely if there is a $C_{A_1A_j}$ as one of the terms, and this can only happen if $n+1-j = 1$ and $k = j$, or if $k = 1$ and $n+1-j = j$. The first implies that $j = n$, which was not one of the original basis elements, but the second option implies that $j = \frac{n+1}{2}$. So if n is odd we must add one more.

Thus we arrive at a total trace of

$$\mathrm{Tr}(\text{conj. by } \rho) = \begin{cases} \frac{n}{2} - 1 = \left\lfloor \frac{n-1}{2} \right\rfloor, & n \text{ even} \\ \left\lfloor \frac{n}{2} \right\rfloor - 1 + 1 = \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n-1}{2} \right\rfloor, & n \text{ odd} \end{cases} = \left\lfloor \frac{n-1}{2} \right\rfloor.$$

□

6.2. **Proofs of Preference Function Properties.** The second part of the appendix consists of several longer proofs that the social preference functions we have been discussing do, in fact, send the correct components of the profile space where we claim they do.

Proof of Proposition 14. Consider that B_X consists of a profile with $n+1-2k$ voters for each ranking with X in kth place. We will look at the BC and KR separately. Recall that the SRSF gives $\sum_v s(v,r)$ points to ranking r, where the sum is over the whole profile (in the case of a differential, we can just subtract the negative v). We will see that each of them sends B_X to a multiple of B_X, which by symmetry will mean that this is the component.

For the Borda Count, since it is a positional scoring rule, for a given r we have that $s(v,r) = (n-1)t(v,r(1)) + (n-2)t(v,r(2)) + \cdots + t(v,r(n-1))$. We will break up the sum over all v into sums over each of the subsets where $v(k) = X$. This gives

$$\sum_{k=1}^{n}\left(\sum_{v(k)=X}(n+1-2k)\left[(n-1)t(v,r(1))+(n-2)t(v,r(2))+\cdots+t(v,r(n-1))\right]\right)$$

This in itself has a high degree of symmetry.

Assume that $X = r(j)$. Consider the terms with k and $n+1-k$. These will have opposite coefficients in sign. Further, since B_X has the same number of voters for *all* rankings with $v(k) = X$, there are the same number of v with $v(k) = X$ and some given $v(i) = r(\ell)$ for $i \neq k, n+1-k$ as there are (negatively) v with $v(n+1-k) = X$ and those same $v(i) = r(\ell)$ for each of $i \neq k, n+1-k$. That means that the terms in $s(v,r)$ corresponding to these will cancel, since it will correspond to $(n-\ell)t(v, r(\ell))$ when $r(\ell) = v(i)$, and we just said there are equal numbers of these for k and $n+1-k$ as long as $i \neq k, n+1-k$. So, for each v such that $v(k) = r(j) = X$ and $v(n+1-k) = r(i)$ (where obviously $i \neq j$), we get

$$(n+1-2k)\left[(n-j)t(v,r(j))+(n-i)t(v,r(i))\right]\ ,$$

which is okay since when $k = 1$ we correctly get 0 as the inside coefficient in $s(v,r)$. Once we sum this up and substitute in the Borda Count values of $t(v,r(j)) = t(v,X) = \frac{n-k}{n-1}$ and $t(v,r(i)) = \frac{n-(n+1-k)}{n-1} = \frac{k-1}{n-1}$, we get

$$\sum_{k=1}^{n}\left(\sum_{v(k)=X=r(j),\,v(n+1-k)=r(i)}(n+1-2k)\left[(n-j)\frac{n-k}{n-1}+(n-i)\frac{k-1}{n-1}\right]\right)\ .$$

Now consider that there are $(n-1)!$ different v such that $v(k) = X$, and hence $(n-2)!$ different v such that $v(k) = X$ *and* $v(n+1-k) = r(i)$ in the above sum. Then we get

$$\sum_{k=1}^{n}(n+1-2k)\left((n-1)!\,(n-j)\frac{n-k}{n-1}+(n-2)!\sum_{i\neq j}(n-i)\frac{k-1}{n-1}\right)\ .$$

In fact, a little clearing of denominators yields

$$(n-2)!\sum_{k=1}^{n}(n+1-2k)\left((n-j)(n-k)+\sum_{i\neq j}(n-i)\frac{k-1}{n-1}\right)\ .$$

The reader will notice that the sum only depends on j, as one would hope. If we increase j by one, the difference between two of these scores is

$$(n-2)!\sum_{k=1}^{n}(n+1-2k)\left((n-j)(n-k)+\sum_{i\neq j}(n-i)\frac{k-1}{n-1}\right)-$$
$$(n-2)!\sum_{k=1}^{n}(n+1-2k)\left((n-(j+1))(n-k)+\sum_{i\neq j+1}(n-i)\frac{k-1}{n-1}\right)$$

which can be simplified to a formula *not* depending on j, as needed:

$$(n-2)!\sum_{k=1}^{n}(n+1-2k)\left((n-k)-\frac{k-1}{n-1}\right)=(n-2)!\frac{n^2(n+1)}{6}\ .$$

Similarly, we need that $j=1$ and $j=n$ are opposites, and indeed

$$(n-2)!\sum_{k=1}^{n}(n+1-2k)\left((n-1)(n-k)+\sum_{i\neq 1}(n-i)\frac{k-1}{n-1}\right)+$$
$$(n-2)!\sum_{k=1}^{n}(n+1-2k)\left((n-n)(n-k)+\sum_{i\neq n}(n-i)\frac{k-1}{n-1}\right)$$

simplifies to

$$(n-2)!\sum_{k=1}^{n}(n+1-2k)\left((n-1)(n-k)+\frac{k-1}{n-1}\left((n-1)+(n-n)+2\sum_{i=2}^{n-1}n-i\right)\right)=$$
$$(n-2)!\sum_{k=1}^{n}(n+1-2k)\left((n-1)(n-k)+\frac{k-1}{n-1}\left(\frac{n(n-1)}{2}+\frac{(n-1)(n-2)}{2}\right)\right)=$$
$$(n-2)!(n-1)^2\sum_{k=1}^{n}(n+1-2k)=0\,.$$

□

Proof of Proposition 15. For the Kemeny Rule, recall that $s(v,r)=\sum_{a,b\in A}\delta(v,r,a,b)$, where δ is 1 if $a\succ b$ by both rankings v and r, and is 0 otherwise; clearly this relies only on pairwise information. Let us see where it sends the (new) Condorcet components.

Using the notation above for $\{XYi\}$ we see that for a given ranking r (with $r\in\{XYj\}$) the score for r is

$$\sum_{i=1}^{n-1}(n-2i)\sum_{v\in\{XYi\}}\sum_{a,b\in A}\delta(v,r,a,b)\,.$$

This depends only on j since the sums are over all $v\in\{XYi\}$. For each $v\in\{XYi\}$ such that $a\succ_v b$ and $a\succ_r b$, reversing r will cause δ to go from 1 to 0, but will send δ from 0 to 1 for the reversal of v; since this reversal is in $\{XY(n-i)\}$, the score for the reversal of r is

$$\sum_{i=1}^{n-1}(n-2i)\sum_{v\in\{XY(n-i)\}}\sum_{a,b\in A}\delta(v,r,a,b)=\sum_{k=1}^{n-1}(2k-n)\sum_{v\in\{XYk\}}\sum_{a,b\in A}\delta(v,r,a,b)$$

which is a change of sign, as expected.

Furthermore, if r changes from $\{XYj\}$ to $\{XY(j+1)\}$ via a one-position swap, then all δ values will be the same except ones which are about that pair. So one could say that each v with that pair as in r loses 1, while each with the pair as in r' gains one.

If such a swap changes things, it must clearly involve X or Y; we will suppose it is $Y\_$ changing to $\_Y$ (other possibilities are very similar). The number of potential places for the candidate $\_$ to occur *after* Y in a ranking in the set $\{XYi\}$ varies from zero to $n-2$, but there will be two potential ones of type $n-1-i$ (such as, for $i=2$, $X\_Y\_\ldots\_$ and $\_Y\_\ldots\_X$). Hence there are, for $\{XYi\}$, $n-1-i+\sum_{i=0}^{n-2}i=$

$n-1-i+\frac{(n-1)(n-2)}{2}$ possibilities out of a total of $n(n-2)$ total; the difference is

$$n-1-i+\frac{(n-1)(n-2)}{2}-\left(n(n-2)-\left(n-1-i+\frac{(n-1)(n-2)}{2}\right)\right)=$$

$$2\left(n-1-i+\frac{(n-1)(n-2)}{2}\right)-n^2+2n=2n-2-2i+(n-1)(n-2)-n^2+2n=n-2i$$

The other $n-3$ spots will have $(n-3)!$ different possibilities, so there are $(n-2i)(n-3)!$ net rankings v in $\{XYi\}$ which will lose $\delta=1$ (which, for i for which this is negative, correspond to gaining $\delta=1$). Thus the difference in the scores given by

$$\sum_{i=1}^{n-1}(n-2i)\sum_{v\in\{XYi\}}\sum_{a,b\in A}\delta(v,r,a,b)$$

will be

$$\sum_{i=1}^{n-1}(n-2i)\left[(n-2i)(n-3)!\right]=(n-3)!\sum_{i=1}^{n-1}(n-2i)^2=\frac{n!}{3}$$

□

Proof of Proposition 24. We use the same decomposition as above for the Borda Count, over $v(k)=X=r(j)$. As above,

$$\sum_{k=1}^{n}\left(\sum_{v(k)=X}(n+1-2k)\sum_{a,b\in A}\delta(v,r,a,b)\right).$$

This also only depends on j (here, r is a fixed ranking with $r(j)=X$, but nothing else known) because of the sum over all v with each $v(k)=X$. However, it is useful to focus on a specific r for proving this sends Borda to Borda.

First let us observe what happens to the score when r is reversed. For each v such that $a\succ_v b$ and $a\succ_r b$, we will get $+(n+1-2k)$, depending on $v(k)=X$. But if r is reversed, these go away, and the reversal of v will have $\delta=1$. This will exactly give the negative of the original score, because if $v(k)=X$, then the reversal v' has $v'(n+1-k)=X$, which means it will contribute $+(n+1-2(n+1-k))=-n-1+k=-(n+1-k)$ to the score.

We also need to have a fixed change in score when r changes. Let r' be the same ranking as r but with $r'(j+1)=X$. As with the C_{XY} components, we will suppose the change is $X\_$ changing to $\_X$. The number of potential places for the candidate $\_$ to occur *after* X in a ranking in the set $\{v(i)=X\}$ is $n-i$, and the number of places before is $i-1$, so the difference is $n+1-2i$. For the remaining spots there are $(n-2)!$ possibilities, so there are $(n+1-2i)(n-2)!$ net rankings v in $\{v(i)=X\}$ which will lose $\delta=1$ (which, for i for which this is negative, correspond to gaining $\delta=1$).

Thus the difference in the scores given by

$$\sum_{k=1}^{n}\left(\sum_{v(k)=X}(n+1-2k)\sum_{a,b\in A}\delta(v,r,a,b)\right)$$

will be

$$\sum_{k=1}^{n}\left((n+1-2k)^2(n-2)!\right)=(n-2)!\sum_{k=1}^{n}(n+1-2k)^2=\frac{(n+1)!}{3}$$

□

Proof of Proposition 20. Let us see what happens to the profile $\mathbf{p}$ which has value $\mathbf{p}(v) = 1$ for $v(j) = X$ and zeros otherwise. In this case,

$$\sum_{k=1}^{n}\left(\sum_{v(k)=X}\left(\sum_{i=1}^{n-1}(n-i)t(v,r(i))\right)\right) = \sum_{v(j)=X}\left(\sum_{i=1}^{n-1}(n-i)t(v,r(i))\right)$$

Now suppose that $r(\ell) = X$; then it makes sense to rewrite this as

$$\sum_{v(j)=X}\left((n-\ell)t(v,r(\ell)) + \sum_{i=1,\, i\neq\ell}^{n-1}(n-i)t(v,r(i))\right).$$

Call the weighting vector w. Then note that in the second sum inside the parentheses, for a given i, $v(k) = r(i)$ the same number of times (to be precise, $(n-2)!$ times), other than $v(j)$, of course. So we can rewrite this

$$\sum_{v(j)=X}\left((n-\ell)w(j) + \sum_{v(j)=X}\sum_{i=1,\, i\neq\ell}^{n-1}(n-i)t(v,r(i))\right) =$$

$$(n-1)!(n-\ell)w(j) + \sum_{i=1,\, i\neq\ell}^{n-1}(n-i)(n-2)!\sum_{k=1,\, k\neq j}^{n} w(k) =$$

$$(n-2)!\left[(n-1)(n-\ell)w(j) + \sum_{k=1,\, k\neq j}^{n}\sum_{i=1,\, i\neq\ell}^{n-1} w(k)(n-i)\right]$$

It is easy to see that the difference in this caused by changing $r(\ell) = X$ to $r(\ell+1) = X$ is $(n-2)!\left[(n-1)w(j) - \sum_{k=1,\, k\neq j}^{n} w(k)\right]$, which does not depend on ℓ. Likewise, adding the cases for $\ell = 1$ and $\ell = n$ gives

$$(n-2)!\left((n-1)^2 w(j) + \sum_{k=1,\, k\neq j}^{n} w(k)\left(\sum_{i=1}^{n-1} n-i + \sum_{i=2}^{n-1} n-i\right)\right) =$$

$$(n-1)!(n-1)\sum_{k=1}^{n} w(k)$$

so that it goes to the Borda component alone if the weighting vector is sum-zero, otherwise it is 'shifted' by a multiple of the sum of the weights. □

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