A Sampler of the Mathematics of Voting and Choice

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Gordon College

Colby College Mathematics and Statistics Colloquium
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Outline

Fair Division

Apportionment and Redistricting

The Mathematics of Voting

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- If the whole cake is pretty much the same, I'll just give it all to K.
- Now assume the cake is half chocolate and half strawberry, *and* K and M like chocolate and strawberry, respectively, *way* more than the other option.

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- So does your final idea have all three properties?

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Such ideas are quite old; Abraham and Lot (in the Hebrew Bible) even use a simple envy-free procedure to decide where to graze their flocks!

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Let's try this now. (What basic calculus theorem can prove this works?)

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- This is of great interest to economists – and hence strategic considerations are part of any analysis.

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Let's see what the Constitution has to say.

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So now what?

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- Jefferson rounded *down*, which happens to favor large states – like his and Washington's Virginia.

How to distribute influence

This shows some steps of Jefferson's method using some (slightly cooked) data (like that) from the first census.

	A	B	L	M	N	O
1	State	Pop.	New Div. #2	Round Down	New Div. #3	Round Down
2	Connecticut	237,655	6.60152778	6	6.71341808	6
3	Delaware	59,096	1.64155556	1	1.66937853	1
4	Georgia	82,548	2.293	2	2.33186441	2
5	Kentucky	73,677	2.04658333	2	2.08127119	2
6	Maryland	319,728	8.88133333	8	9.03186441	9
7	Massachusetts	475,199	13.1999722	13	13.4237006	13
8	New Hampshire	141,899	3.94163889	3	4.00844633	4
9	New Jersey	184,139	5.11497222	5	5.20166667	5
10	New York	340,241	9.45113889	9	9.61132768	9
11	North Carolina	395,005	10.9723611	10	11.1583333	11
12	Pennsylvania	433,611	12.04475	12	12.2488983	12
13	Rhode Island	69,112	1.91977778	1	1.95231638	1
14	South Carolina	249,073	6.91869444	6	7.03596045	7
15	Vermont	85,341	2.37058333	2	2.41076271	2
16	Virginia	747,550	20.7652778	20	21.1172316	21
17	Total:	3,893,874		100		105
18	House Size:	105				
19	Standard Divisor:	37084.51429	36000		35400	
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  - (This almost happened in 1882, where Alabama would have had more seats in a Congress of 299 reps than one of 300.)

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But it didn't really matter, from a mathematical perspective, because:

- **Fact**: Any method like Jefferson's or Webster's suffers from a big flaw; you can get more (or fewer) representatives than either a straight rounding up *or* rounding down would give!
  - (This happened in 1832, with New York getting 40 seats in Congress even though it 'deserved' 38.6 seats.)
  - The method we use today also can have this happen... though so far, it hasn't!
- **Fact**: Any method like Hamilton's suffers from a big (bigger?) flaw; your state can lose a seat due *only* to Congress getting bigger! (**Not** due to population changes.)
  - (This almost happened in 1882, where Alabama would have had more seats in a Congress of 299 reps than one of 300.)

In the early 1980s, Michael Balinski and Peyton Young *mathematically* unified and proved everything known and observed up to that time. Their work has been quoted in several Supreme Court decisions about this issue.

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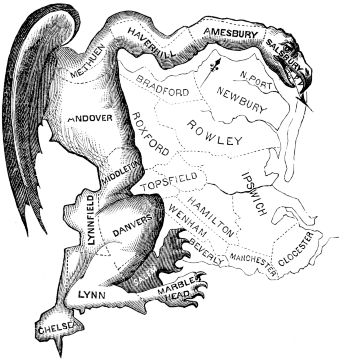

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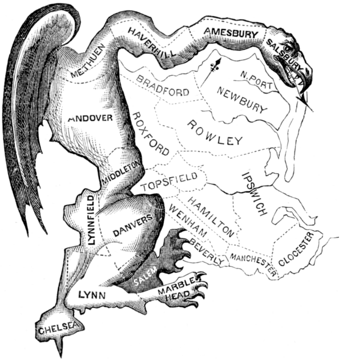

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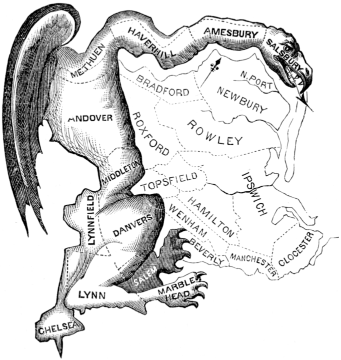

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(There is a lot of undergraduate research in this area.)

Outline

Fair Division

Apportionment and Redistricting

The Mathematics of Voting

Same Election, Different Results

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(Change a few names to get the Minnesota election for governor in 1998, where radio host, wrestler, and small-town mayor Jesse Ventura won.)

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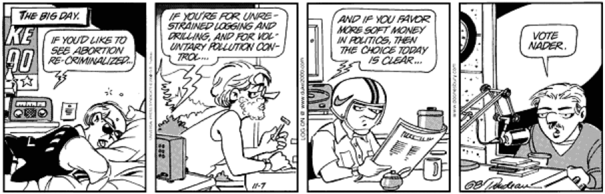

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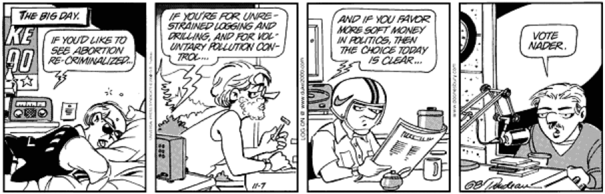

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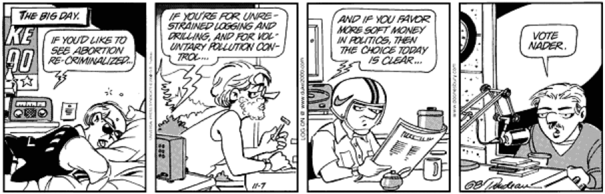

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One could try a different method. . .

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What is going on here?

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But that doesn't stop people from trying to advocate for their preferred method.

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  - That is, when it's been used, it needs special warnings – like to *not* vote for someone just to encourage them!

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  - It's fairly easy to explain – your second (and third and . . .) choices count if your first choice doesn't win.
  - But IRV is susceptible to a truly horrific paradox. It is quite possible for additional campaigning by a candidate to add support, *directly* causing them to do **worse** in the election!

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So it is up to **us** to be informed!

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- We can think of the voters from the popularity contest as the vector $(35, 0, 20, 17, 0, 28)$, where each entry corresponds to an ordering of candidates. (Recall none had G in second place.)
- We can decompose the vector with respect to the following basis:

$$(35, 0, 20, 17, 0, 28) =$$
$$\frac{100}{6}(1, 1, 1, 1, 1, 1) + \frac{10}{6}(1, -1, 1, -1, 1, -1) + \frac{22}{3}(1, 1, 0, -1, -1, 0)$$
$$+ \frac{17}{3}(0, -1, -1, 0, 1, 1) - 8(1, 1, -2, 1, 1, -2) + \frac{26}{3}(-2, 1, 1, -2, 1, 1)$$

- Believe it or not, the coefficients alone tell us *exactly* how it will behave with respect to all the systems we care about.

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There is very interesting combinatorics, propositional logic, and algebra behind analysis of this kind of situation.

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