HOMEWORK HELP FOR MATH 152

So here's the next version of Homework Help!!!

I am going to assume that no one had any great difficulties with the problems assigned this quarter from 4.3 and 4.4. However, if you can't do them or ones like them, you won't be able to do anything else. Clearly Max-Min problems were the most difficult for people, so the most of these are those.

4.5, 28

Two hallways, one 8 feet wide and the other 6 feet wide, meet at right angles. Determine the length of the longest ladder that can be carried horizontally from one hallway into the other.

Here a picture is of course extremely helpful. So draw one with the two hallways intersecting, and then draw a sample ladder (just a straight line, since we have a top view and presumably one would want to hold the ladder on the edge so as to not block even more room) touching the inner corner and the two outside hall walls. How do we set up this problem? The key is to *minimize* the length of this hypothetical ladder! If we do that, then any ladder smaller than that one can definitely make it through; but a longer one will not, because it will get stuck before our *minimum* length. So here, longest actually means shortest. Sort of.

There are several ways to do this. Here is a relatively quick one. The ladder is clearly the hypotenuse of some right triangle; which one? We can let it be the hypotenuse of a triangle with base $x + 8$ and height $y + 6$, since we know that in any case the ladder will stretch across the wide hall, plus some, and the thin hall, plus some different amount. Like I said, drawing it is the only way to really see this. So the length is $\sqrt{(x+8)^2 + (y+6)^2}$. Further, we have another relation between the quantities involved. We look at the triangle formed by the ladder, the lower hallway, and the perpendicular line to the hall; this is similar to the triangle formed by the ladder, the upper hallway, and the perpendicular line to the hall. So by properties of similar triangles, the ratios of similar sides are equal; in this case, $\frac{x}{6} = \frac{8}{y}$, or $xy = 48$. So I replace $y = \frac{48}{x}$.

Now we do the calculus. We want to minimize $\sqrt{(x+8)^2 + (48/x+6)^2}$. The derivative is

$$\frac{x + 8 - (48/x + 6)(48x^{-2})}{\sqrt{(x+8)^2 + (48/x+6)^2}}$$

We merely check when the numerator is zero, since the denominator always is positive. This happens when $x + 8 = \frac{48^2+6(48)x}{x^3}$; we divide both sides by $x + 8$ (since that is definitely not a solution!) and get $x^3 = 6(48) = 288$. Thus the only critical point of interest is when $x = \sqrt[3]{288} = 2\sqrt[3]{36}$, which you can check is in fact a minimum; for this x, $y = 4\sqrt[3]{18}$. Plug these in and get the total ladder length!

4.5, 30

Conical paper cups are usually made so that the depth is $\sqrt{2}$ times the radius of the rim. Show that this design requires the least amount of paper per unit volume.

The key to solving this problem is interpreting the second sentence. What are we being asked? In fact, this is a minimizing problem. Cups are made so that the ratio $\frac{\text{depth}}{\text{radius}} = \sqrt{2}$, and we want to show that this ratio gives us the least paper per unit volume (hence cheapest). And what is 'least paper per unit volume?' One way to interpret this is simply as the minimum of the ratio $\frac{\text{area}}{\text{volume}}$.

So let's use the formulae we have in the book to help us. Let r be the radius of the rim and d be the depth of the cup (you may want to draw this; unfortunately my computer skills aren't quite up to making and inserting such drawings). We have the formulae surface area $A = \pi r\sqrt{r^2 + d^2}$ and volume $V = \frac{1}{3}\pi r^2 d$. We will fix the volume, so as to find the ratio; let it be $\frac{\pi}{3}$. Then $\frac{1}{3}\pi r^2 d = \frac{\pi}{3}$, so $r^2 = \frac{1}{d}$, so the area will be $A = \pi\sqrt{\frac{1}{d}(\frac{1}{d} + d^2)} = \pi\sqrt{\frac{1}{d^2} + d^2}$. We wish to minimize this.

We then do the calculus. Let $f(x) = \pi\sqrt{\frac{1}{d^2} + d^2}$. Then $f'(x) = \frac{\pi}{2}\frac{\frac{-2}{d^3}+1}{\sqrt{\frac{1}{d^2}+d^2}}$. After a little algebra we see that this is zero precisely when $\pi(d^3 - 2) = 0$, or when $d = \sqrt[3]{2}$. When this happens, $r^2 = \frac{1}{\sqrt[3]{2}}$, so $r = 2^{-1/6}$. Thus $\frac{d}{r} = 2^{1/3+1/6} = \sqrt{2}$, which is the desired answer.

4.5, 44

A tapestry 7 feet high hangs on a wall. The lower edge is 9 feet above an observer's eye. How far from the wall should the ovserver stand to obtain the most favorable view? Namely, what distance from the wall maximizes the visual angle of the observer?

This problem has one fundamental fact behind it. This is the fact that an angle is maximized when the tangent of that angle is maximized. This follows from the fact that tangent is an increasing function. Without this, all we could do is maximize the value of tangent for a given distance of the observer, as we shall soon see. But this distance will also maximize the angle itself, since if there was some other distance which maximized the angle, then (as tangent is increasing) tangent would be maximized there, and not the original place. This is difficult to convey without pictures or live explanation, so please come on in if it's still unclear.

So we imagine the situation; the observer looks up from the ground a total of α radians to see the top of the tapestry, 16 feet up, and looks only β radians to see its bottom, 9 feet up. So we would like to maximize $\alpha - \beta$. For now, we can only look at $\tan(\alpha - \beta)$. But as noted above, maximizing this is good enough. Further, we *do* know $\tan\alpha$ and $\tan\beta$ for a given distance from the wall; they are $16/x$ and $9/x$, respectively. Then we use the formula found on page 46 in the text (which one could, however, independently derive); $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$. This gives us $\tan(\alpha - \beta) = \frac{7/x}{1+144/x^2} = \frac{7x}{x^2+144}$.

The derivative of this is $\frac{-7x^2+7(144)}{(x^2+144)^2}$. Since the denominator is always positive, we need merely check when $x^2 = 144$, which is for $x = 12$ (the negative value is meaningless for us). Checking the derivative for nearby points, we see that this is in fact a maximum. So the answer is that the observer should stand twelve feet away. Notice that this does not require us to calculuate either the actual tangent at that point *or* even the angle! We have proved it is a maximum, and that is enough.

4.5, 52

A local bus company offers charter trips to Blue Mountain Museum at a fare of 37 dollars per person if 16 to 35 passengers sign up for the trip. The company does not charter trips for fewer than 16 passengers. The bus has 48 seats. If more than 35 passengers sign up, then the fare for every passenger is reduced by 50 cents for each passenger in excess of 35 that signs up. Determine the number of passengers that generates the greatest revenue for the bus company.

This little puzzle has two interesting facets; one, the function describing the revenue has two parts, and, second, the only allowed solutions are positive integers. But we can still look for critical points. The pertinent function is

$$f(x) = \begin{cases} 37x, & 16 \leq x \leq 35 \\ (37 - .5(x - 35))x, & 36 \leq x \leq 48 \end{cases}$$

The critical points are thus the (possible) discontinuity at $x = 35, 36$ and and critical points of the second case, since $(37x)' = 37 \neq 0$. But the second expression has zero derivative only for $x = 54.5$; in fact, the function is increasing throughout that time. Then we merely note that $f(35) < f(36)$ (as you can check for yourself) and that means the endpoint is the absolute maximum. So having 48 passengers gives the best revenue. Though not necessarily the best ride, depending on the shocks in the bus.

4.6, 21

Find all pertinent information about $f(x) = x^3 - 9x$.

We know that it factors: $x^3 - 9x = x(x - 3)(x + 3)$. So it has zeros at ± 3 and 0. The derivative is $3x^2 - 9$, which is zero at $\pm\sqrt{3}$. Using the corollary to the intermediate value theorem, since $f'(-3) > 0$, $f'(0) <= 0$ and $f'(3) > 0$, we know that f is increasing on $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$ and decreasing on $(-\sqrt{3}, \sqrt{3}$. Thus it has a local maximum at $-\sqrt{3}$ and a local minimum at $\sqrt{3}$. Further, we can examine the second derivative $f''(x) = 6x$. This is positive for $x > 0$ and negative for $x < 0$. So f is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$. Thus $(0, 0)$ is a point of inflection.

4.6, 28

Find all pertinent information about $f(x) = \begin{cases} 2x + 4, & x \leq -1 \\ 3 - x^2, & x > -1 \end{cases}$.

First we check to see if it is continuous and/or differentiable at $x = -1$, since that is a change in definition. We have $\lim_{x\to -1^+} f(x) = 2 = f(-1)$, so it is continuous. Further, $\lim_{x\to -1^+} f'(x) = 2 = \lim_{x\to -1^-} f(x)$, so it is even differentiable everywhere. Clearly the only zeros of f are at $x = -2$ and $x = \sqrt{3}$, since the other possibility of $x = -\sqrt{3}$ is not zero for this f. The derivative is $f'(x) = 2$ for $x \leq -1$, and $f'(x) = -2x$ for $x > -1$, so the function is increasing for $x < 0$ and decreasing for $x > 0$. On the linear part it is neither concave up nor concave down, while $f''(x) = -2$ for $x > -1$ so it is concave down there. Hence there is no point of inflection.

4.7, 2

Look on page 246 of your text. Answer the questions given.

We see that $\lim_{x\to\infty} g(x) = d$ and $\lim_{x\to b^+} g(x) = C$. There are two vertical asymptotes, namely $x = a$ and $x = b$. We have already noted the horizontal asymptote $y = d$. There appears to be a vertical tangent at $x = p$, and a vertical cusp at $x = q$.

4.7, 32

Determine whether or not the graph of $f(x) = \sqrt{4 - x^2}$ has a vertical tangent or a vertical cusp at $c = 2$.

This was quite a tricky problem for many of you. The best answer I saw was "it's half a tangent." Why is this? Well, for one thing f isn't even defined for $x > 2$. Now this is no barrier for it to be a vertical asymptote, but the vertical cusps and tangents ordinarily require a little more. You can check that $f'(x) = -\frac{x}{\sqrt{4-x^2}}$ has a limit $\lim_{x\to 2^-} f'(x) = -\infty$; but there isn't any function left over to have $\lim_{x\to 2^+} f'(x) = -\infty$ for a vertical tangent or $\lim_{x\to 2^+} f'(x) = \infty$ for a vertical cusp! Incidentally, the graph of this is just a semicircle of radius two, the half above the x-axis.

4.8, 55

A function f is continuous, differentiable for all $x \neq 0$, and $f(0) = 0$. Find relevant information based on the graph of the derivative of f on page 255.

Since the derivative is positive on $(-\infty, -1) \cup (0, 1) \cup (3, \infty)$, that is also where f is increasing; hence f is decreasing on $(-1, 0) \cup (1, 3)$. The critical numbers are $x = -1, 0, 1$ and 3. I cannot sketch the graph of f'', but again note that f' is increasing on $(-\infty, -3) \cup (2, \infty)$, so this is where f is concave up, and similarly f is concave down on $(-3, 0) \cup (0, 2)$. Finally, the horizontal asymptote visible for f' indicates there is also one there for f, though of course we cannot tell what value $\lim_{x\to-\infty} f(x)$ takes on.

4.8, 57

Show that the lines $y = (b/a)x$ and $y = -(b/a)x$ are oblique asymptotes of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

What does it mean to be an asymptote? That means that the function approaches a given line arbitrarily close, the further out one goes. In particular, we can say an asymptote to $L(x)$ exists if $\lim_{x\to\infty} f(x) - L(x) = 0$. So then the question here is, are the given lines asymptotes? Since the hyperbola is clearly symmetric about the x-axis, I will only consider positive y; this enables me to rewrite the equation as $y = \sqrt{b^2(x^2/a^2 - 1)}$.

So $\lim_{x\to\infty} f(x) - L(x) = \lim_{x\to\infty} \sqrt{b^2(x^2/a^2 - 1)} - (b/a)x$. We can rewrite this as (assuming $a, b > 0$) $\lim_{x\to\infty}(b/a)(\sqrt{x^2 - a^2} - x)$. However, as x gets exceedingly large, $\sqrt{x^2 - a^2}$ does indeed approach x, since the a^2 becomes relatively inconsequential (or use the theorems on limits commuting with continuous functions to see this more rigorously). So the whole limit becomes zero, as desired. The other cases (where we don't have $a, b > 0$ or we go to $-\infty$ are similar.

5.1, 12

Given that $P = \{x_0, x_1, \dots, x_n\}$ is an arbitrary partition of $[a, b]$, find $L_f(P)$ and $U_f(P)$ for $f(x) = x + 3$. Use this answer to evaluate $\int_a^b f(x)dx$.

The minimum value of $f(x)$ on $[x_{i-1}, x_i]$ is clearly $x_{i-1} + 3$, and the maximum value is $x_i + 3$ on the same interval, since the function is increasing. So our sums look like

$$U_f(P) = (x_1 - x_0)(x_1 + 3) + (x_2 - x_1)(x_2 + 3) + \cdots + (x_n - x_{n-1})(x_n + 3) \text{ and}$$
$$L_f(P) = (x_1 - x_0)(x_0 + 3) + (x_2 - x_1)(x_1 + 3) + \cdots + (x_n - x_{n-1})(x_{n-1} + 3).$$

These sums collapse somewhat, because $3(x_1-x_0)+3(x_2-x_1)+\cdots+3(x_n-x_{n-1}) = 3(x_n - x_0)$. So we have

$$U_f(P) = (x_1 - x_0)(x_1) + \cdots + (x_n - x_{n-1})(x_n) + 3(b - a) \text{ and}$$
$$L_f(P) = (x_1 - x_0)(x_0) + \cdots + (x_n - x_{n-1})(x_{n-1}) + 3(b - a).$$

We evaluate the integral by considering what happens to the lower and upper sums. I refer you at this point to the file bookdiff.pdf, which describes in greater detail exactly what the difference is. In either case, of course, we get $3(b - a) + (b^2 - a^2)/2$.

5.1, 18

Let f be continuous on $[a, b]$, let $P = \{x_0, x_1, \dots x_n\}$ be a partition of $[a, b]$, and let $S^*(P)$ be any Riemann sum generated by P. Show that

$$L_f(P) \le S^*(P) \le U_f(P).$$

For any subinterval $[x_{i-1}, x_i]$, we know that the value for L_f is $m_i = \min_{[x_{i-1},x_i]} f(x)$, and the value for U_f will be $M_i = \max_{[x_{i-1},x_i]} f(x)$. Now the value for any old Riemann sum will be $f(x_i^*)$, which by definition of M and m has the property $M_i \ge f(x_i^*) \ge m_i$. Thus $(\Delta x_i)M_i \ge (\Delta x_i)f(x_i^*) \ge (\Delta x_i)m_i$ as well. But we add up a finite number of such things for all three types of sums! So

$$(\Delta x_1)M_1+\cdots+(\Delta x_n)M_n \ge (\Delta x_1)f(x_1^*)+\cdots+(\Delta x_n)f(x_n^*) \ge (\Delta x_1)m_1+\cdots+(\Delta x_n)m_n$$

which is just $L_f(P) \le S^*(P) \le U_f(P)$.

5.1, 23,24 and 28

Assume that f and g are continuous, $a < b$, and $\int_a^b f(x)dx > \int_a^b g(x)dx$. Which statements hold for all partitions P of $[a, b]$?

We have $L_g(P) < U_f(P)$, because $L_g(P) < \int_a^b g(x)dx < \int_a^b f(x)dx < U_f(P)$ for any partition (by definition of the upper and lower sums). But $L_g(P) < L_f(P)$ is not necessarily true; for instance, for the partition $P = \{a, b\}$, if $f(a) = g(a)$ and both functions are increasing, will give lower sums which are equal. Finally, by the same reasoning, except using $f(b) = g(b)$, we don't necessarily have $U_g(P) < \int_a^b f(x)dx$ all the time.

5.2, 12

For $F(x) = \int_1^x \sin \pi t \, dt$, compute $F'(-1)$, $F'(0)$, $F'(\frac{1}{2})$, and $F''(x)$.

By the first fundamental theorem of calculus, $F'(x) = \sin \pi t$. So the answers are 0, 0, 1 and $\pi \cos \pi t$.

5.2, 17

Find the derivative of $F(x) = \int_0^{x^3} t \cos t \, dt$.

We use the chain rule. Then, since $\frac{d}{dx} x^3 = 3x^2$ and by the first fundamental theorem of calculus, we get $x^3 \cos x^3 (3x^2) = 3x^5 \cos x^3$.

5.3, 48

Evaluate the integrals

$$\int_{-4}^{2} (2x+3) dx \text{ and } \int_{-4}^{2} |2x+3| dx$$

By the second fundamental theorem of calculus, the first integral is $(x^2+3x)_{-4}^2 = 6$. The second integral needs to be split up into a couple pieces. Since $|2x+3| = 0$ for $x = -\frac{3}{2}$, we compute $\int_{-4}^{-3/2} -(2x+3)dx + \int_{-3/2}^{2} (2x+3)dx$ recalling the definition of absolute value. By the same theorem, we compute $-(x^2+3x)_{-4}^{-3/2} + (x^2+3x)_{-3/2}^{2}$, which simplifies to $9/4 + 4 + 6 + 12 + 9/4 = 26\frac{1}{2}$.

5.4, 26

Find the area bounded by $y = x + 1$, $y = \cos x$, and $x = \pi$.

It is easy to see that $x + 1 > \cos x$ on the interval in question, namely $[0, \pi]$. So we simply calculate $\int_0^{\pi} x + 1 - \cos x \, dx$, which is (by the second fundamental theorem of calculus) $(-\sin x + \frac{x^2}{2} + x)_0^{pi}$, which comes out to be $\frac{\pi^2}{2} + \pi$.