1. Which elementary matrix represents adding two times the first row to the third row? multiplying the second row by 3? switching the second and third rows? Use them on a matrix of your choice.

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
2 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

Since I don’t know which matrix you chose, you will need to use them on the matrix of your choice yourself.

2. Find the row space, column space, and kernel of the following matrix.

\[
\begin{pmatrix}
2 & 4 & 6 & 18 \\
4 & 5 & 6 & 24 \\
3 & 1 & -2 & 4 \\
\end{pmatrix}
\]

Now find a basis for each of those spaces. What are the dimensions of each of the spaces? Confirm the so-called rank-nullity theorem using this information.

We begin, as always, with row reduction. The following sequence of operations will work: divide the first row by 2, subtract the third from the second, subtract the first from the second, subtract three times the first from the third, subtract the second from the first, add three times the second to the third, subtract twice the third row from the second, divide the second row by 3, and switch the second and third rows. This gives me

\[
\begin{pmatrix}
1 & 0 & -2 & -2 \\
0 & 1 & 4 & 10 \\
0 & 0 & 1 & 3 \\
\end{pmatrix}
\]

So the row space is \( \{t, u, -2t + 4u + v, -2t + 10u + 3v | t, u, v \in \mathbb{R}\} \), the column space is \( \{s - 2u - 2v, t + 4u + 10v, u + 3v | s, t, u, v \in \mathbb{R}\} \), and the null space is \( \{-4v, 2v, -3v, v | v \in \mathbb{R}\} \). By our methods for finding bases, we have (for the row, column, and null spaces respectively):

\[
\begin{pmatrix}
1 \\
0 \\
-2 \\
\end{pmatrix},
\begin{pmatrix}
0 \\
1 \\
4 \\
10 \\
\end{pmatrix},
\begin{pmatrix}
-4 \\
2 \\
-3 \\
1 \\
\end{pmatrix}
\]

3. Find the determinant of

\[
\begin{pmatrix}
2 & 4 & 6 & 18 \\
4 & 5 & 6 & 24 \\
3 & 3 & 3 & 15 \\
1 & 2 & -1 & 0 \\
\end{pmatrix}
\]

Is this an invertible matrix? Describe the set of solutions to the homogeneous system of linear equations defined by it.
To solve, first row-reduce a little bit. Divide the first row by 2, divide the third row by 3 (which changes the determinant by 1/6), then subtract four times the first row from the second (which doesn’t change the determinant), and divide the second row by -3 (which changes the determinant by another -1/3). Then subtract the third row from the second, and finally the second row from the first. The first row now has all zeros. Hence the determinant is zero. So it is not invertible. Finally, some more reduction gives the reduced echelon form

\[
\begin{bmatrix}
1 & 0 & 0 & 13/4 \\
0 & 1 & 0 & -1/2 \\
0 & 0 & 1 & 9/4 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

which shows that the solution set is \(\{(−13v/4, v/2, −9v/4, v) | v ∈ \mathbb{R}\}\).

4. Find an equation for the plane perpendicular to the vector \((-1, 2, 8)\) and through the point \((4, 2, 0)\).

First note that the plane perpendicular to the vector in question, but through the origin, can be considered as vectors (not just points, since through the origin), so we may just think of it as all vectors perpendicular to the given vector. Since dot product zero means perpendicular, the equation for this would be \(-x + 2y + 8z = 0\). However, we need to include the point \((4, 2, 0)\). Hence we will first move the plane we want down to the origin (so its points become vectors); the only point we have in it is \((4, 2, 0)\), so we let its points be \((x - 4, y - 2, z)\). Then applying the same dot product criterion gives \(- (x - 4) + 2(y - 2) + 8z = 0\), or \(-x + 2y + 8z = 0\) - the same equation! So the plane we want contains the origin as it stands.

5. Use Gram-Schmidt to find an orthogonal basis to the subspace of \(\mathbb{R}^3\) spanned by \((2, 3, 4)\) and \((1, 2, -1)\).

Let the first vector be \(\vec{u}\) and the second be \(\vec{v}\). Then we just need to use the projection construction once! So find \(\vec{u} - \text{Proj}_\vec{v}\vec{u}\). That will be

\[
\begin{pmatrix}
2 \\
3 \\
4
\end{pmatrix} - \begin{pmatrix}
2 \\
3 \\
4
\end{pmatrix} \cdot \begin{pmatrix}
1 \\
2 \\
-1
\end{pmatrix} \begin{pmatrix}
1 \\
2 \\
-1
\end{pmatrix}
\]

which gives the vector \((4/3, 5/3, 14/3)\). So this vector and \((1, 2, -1)\) is an orthogonal basis for this subspace.

6. Is the set of vectors \(\{5t + 8v, 3t, 9v + 1\}\) a subspace of \(\mathbb{R}^3\)? If not, why not?

No. The zero vector is not in it, since if it were, \(t = 0\) from the \(y\)-coordinate, which implies \(v = 0\) from the \(x\)-coordinate, which implies the \(z\)-coordinate is 1, which is not the origin - contradiction.

7. Is the set of vectors \((1, 1, 1, 1), (-1, -1, 1, 1), (1, -1, -1, 1)\) a mutually orthogonal set? Are the vectors linearly independent?

Check the three dot products! They are all zero, so it is a mutually orthogonal set. There is no need for any more work to show linear independence by a fact from Chapter 9.1 which was covered in class (orthogonal sets are linearly independent).