MATH 195
SECOND MIDTERM EXAM SOLUTIONS

Each statement is true or false. Write “T” or “F” in each blank.

_____ If a 4x3 matrix has rank 3, the kernel will have dimension 2.
False; here \( n = 4 \) and the rank is 3, so the kernel has dim \( 4-3=1 \).

_____ The set of vectors \( \{(1, 1), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{3}, \frac{1}{3})\ldots\} \) is a vector space.
No; for instance, the zero vector is not in it.

_____ If the matrix \( A \) is symmetric, then the matrix \( A^T \) is square.
True, since the number of rows and columns must be the same for the transpose to equal the original matrix.

_____ The distance between \((5, 2, 1)\) and \((2, -2, 1)\) is 5.
Using the 3-dimensional Pythagorean theorem, the answer is yes.

_____ If \( \vec{v} \in \mathbb{R}^3 \), \( \vec{v} \times \vec{v} = \vec{0} \).
True (compute the cross product).

_____ The maximum rank a 5x4 matrix can have is 4.
True; rank 5 is impossible, since you can only have as many nonzero rows as you have leading ones, each of which are in a different column.

_____ If \( A \) is a 26 \( \times \) 26 matrix with determinant 3, the rank of \( A \) is 23.
False; since the determinant is not zero, it is invertible and has rank 26.

_____ If we project \((1, 2)\) onto \((-1, 3)\), we get \((-\frac{1}{2}, \frac{3}{2})\).
True (do the calculation using the definition of projection).

Next question is short answer.

What is the inverse matrix to \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) 

We know from class the answer is \( \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \); you can check by multiplying them.

Is the set \( \{(1, 2, 3, 5), (2, 3, 5, 8), (3, 5, 8, 13)\} \) a basis for \( \mathbb{R}^4 \)?
No; for instance, \( \mathbb{R}^4 \) has dimension 4, but there are only three vectors in the set.
Also, they are not linearly independent.

Given that all solutions to

\[
\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]
look like \((2z, -z)\), find all solutions to
\[
\begin{pmatrix}
1 & 2 \\
2 & 4
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix} =
\begin{pmatrix}
3 \\
6
\end{pmatrix}.
\]

One way to do this is to recall that solutions to the nonhomogeneous system look like solutions of the homogeneous one, but moved by a solution to the nonhomogeneous one (provided one exists). A solution to the nonhomogeneous system is \((1, 1)\), so \((2x + 1, -x + 1)\) will suffice (for all \(x \in \mathbb{R}\)).

What is the determinant of the matrix
\[
\begin{pmatrix}
2 & 0 & 8 & 3 & 0 & 6 \\
0 & 1 & 2 & \pi & e & -1 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & \sqrt{48} & 9 \\
0 & 0 & 0 & 0 & 3 & 1 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{4}
\end{pmatrix}.
\]

Since it is upper triangular, you can just multiply the diagonals, which yields the answer 1.

Consider the matrix multiplication
\[
\begin{pmatrix}
2 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
3 & 6 \\
83 & 96
\end{pmatrix} =
\begin{pmatrix}
6 & 12 \\
83 & 96
\end{pmatrix}.
\]

If the determinant of the second matrix is \(a\), what is the determinant of the third matrix?

Since the determinant of a product is the product of the determinants, and the determinant of the first matrix is 2 (direct computation), the desired determinant is \(2a\).

Compute
\[
\begin{pmatrix}
8 \\
6
\end{pmatrix} \cdot \begin{pmatrix}
-3 \\
1
\end{pmatrix}.
\]

Well, \(8(-3) + 6(-1) + 5(3) = -15\).

Compute
\[
\begin{pmatrix}
8 \\
6
\end{pmatrix} \times \begin{pmatrix}
-3 \\
3
\end{pmatrix}.
\]

The desired vector will be \(\begin{pmatrix}
\text{det second and third row} \\
\text{det third and first row} \\
\text{det first and second row}
\end{pmatrix}\), which is \(\begin{pmatrix}
23 \\
-39 \\
10
\end{pmatrix}\).

Write the vector \((3, 2, 1, 1)\) as a linear combination of the vectors \((6, 2, 1, 1), (3, 0, 0, -1), (0, 0, 0, 1)\).

For instance, \((3, 2, 1, 1) = (6, 2, 1, 1) - (3, 0, 0, -1) - (0, 0, 0, 1)\).

Define what it means for \(\{\vec{u}, \vec{v}, \vec{w}\}\) to be linearly independent.

\(\{\vec{u}, \vec{v}, \vec{w}\}\) is linearly independent if whenever \(a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}\), then \(a = b = c = 0\).

Show that the set of vectors
\[
\begin{pmatrix}
12 \\
3 \\
4
\end{pmatrix},
\begin{pmatrix}
4 \\
3 \\
12
\end{pmatrix},
\begin{pmatrix}
20 \\
3 \\
-4
\end{pmatrix}
\]
are linearly dependent in any way you wish.
One easy way to do this is to show that the rank of the matrix

\[
\begin{pmatrix}
12 & 4 & 20 \\
3 & 3 & 3 \\
4 & 12 & -4 \\
\end{pmatrix}
\]

is not 3 (by a theorem). In fact, the reduced echelon form of the matrix is

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 0 \\
\end{pmatrix}
\]

so it has rank two.

Next question.

Consider the matrix

\[
A = \begin{pmatrix} 7 & 14 \\ -16 & -1 \\ 2 & 4 \\ -5 & 1 \end{pmatrix}
\]

Give bases for the row space, column space, and kernel of \( A \). Confirm that the so-called Dimension Theorem about the dimension of the null space and the rank of a matrix holds in this case by simple addition.

First, we row reduce \( A \) - an all important step. I will leave it to you to show that in fact the reduced echelon form is

\[
\begin{pmatrix}
1 & 2 & 0 & -7 \\
0 & 0 & 1 & -3 \\
\end{pmatrix}
\]

Then, since the row space does not change under these operations, a basis for it is \((1,2,0,-7), (0,0,1,-3)\). Similarly, for the column space, using the criterion of taking the corresponding columns with leading 1s in the original matrix, we get \( \{(7,2), (-16,-5)\} \) (although since the column space is actually easily seen to be \( \mathbb{R}^2 \), the standard basis works too). What about the kernel? First we back-substitute to find the kernel; we let \( w \) and \( y \) be free variables, in which case \( z - 3w = 0 \) and \( x + 2y - 7w = 0 \). Then the space is \( \{(-2y + 7w, y, 3w, w) \mid y, w \in \mathbb{R} \} \). So we just abstract the basis vectors from that parametrization, which are \( \{(-2,1,0,0), (7,0,3,1)\} \). Then it is clear that the dimension of the kernel (2) plus the rank (2) is the number of variables (4), which verifies the dimension theorem.

Another question.

Find the determinant of the following matrix. Please use at least two row-reductions before using any cofactor expansion, and make sure to take that into account when finding the final answer.

\[
\begin{pmatrix}
2 & -3 & -3 & 7 \\
-2 & 5 & -2 & -5 \\
2 & -3 & 1 & 11 \\
4 & 0 & 10 & 13 \\
\end{pmatrix}
\]

I did the following series of operations, though of course any two you did would be fine. Notice that all mine change the determinant by multiplication by 1 (i.e. not at all).

\[
\begin{pmatrix}
2 & -3 & -3 & 7 \\
-2 & 5 & -2 & -5 \\
2 & -3 & 1 & 11 \\
4 & 0 & 10 & 13 \\
\end{pmatrix} \rightarrow \begin{pmatrix}
2 & -3 & -3 & 7 \\
0 & 2 & -1 & 6 \\
0 & 2 & -1 & 6 \\
0 & 2 & -1 & 6 \\
\end{pmatrix} \rightarrow \begin{pmatrix}
2 & -3 & -3 & 7 \\
0 & 2 & -5 & 2 \\
0 & 2 & -1 & 6 \\
0 & 2 & -1 & 6 \\
\end{pmatrix} \rightarrow \begin{pmatrix}
2 & -3 & -3 & 7 \\
0 & 2 & -1 & 6 \\
0 & 2 & -1 & 6 \\
0 & 2 & -1 & 6 \\
\end{pmatrix} \rightarrow \begin{pmatrix}
2 & -3 & -3 & 7 \\
0 & 2 & -1 & 6 \\
0 & 2 & -1 & 6 \\
0 & 2 & -1 & 6 \\
\end{pmatrix}
\]
\[
\begin{pmatrix}
2 & -3 & -3 & 7 \\
0 & 2 & -5 & 2 \\
0 & 2 & -1 & 6 \\
0 & 6 & 16 & -1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
2 & -3 & -3 & 7 \\
0 & 0 & -4 & -4 \\
0 & 2 & -1 & 6 \\
0 & 6 & 16 & -1
\end{pmatrix}
\]

Then I calculated the determinant using the cofactor expansion down the first column. So the determinant is

\[
2 \begin{pmatrix}
0 & -4 & -4 \\
2 & -1 & 6 \\
6 & 16 & -1
\end{pmatrix}
\]

which is \(2(-2 \det \begin{pmatrix}
-4 & -4 \\
16 & -1
\end{pmatrix} + 6 \det \begin{pmatrix}
-4 & -4 \\
-1 & 6
\end{pmatrix})\), which simplifies to \(-4(4 + 64) + 12(-24 - 4) = -608\).

Final question.
Consider the points \((3, 2)\) and \((-2, 7)\), and the line \(\ell\) which they define in \(\mathbb{R}\). Find a single equation (i.e. not parametric representation) for the line perpendicular to \(\ell\) through the point \((5, 1)\) using the dot product.

First, we find a representation for \(\ell\). We do this the usual way; subtract a point from the other, parametrize, and move back. So \((-2, 7) - (3, 2) = (-5, 5)\), giving a line through the origin \(\{t(-5, 5) | t \in \mathbb{R}\}\). Moving it back gives \(\{t(-5, 5) + (3, 2) | t \in \mathbb{R}\}\). But we want a line perpendicular to this. If it were through the origin, we would simply dot product the direction vector \((-5, 5)\) with \((x, y)\) and set it to zero; however, we desire a line through \((5, 1)\), so we first move all points on this line so it is through the origin, i.e. so that \((x - 5, y - 1)\) is perpendicular to \((-5, 5)\). But then the dot product gives us \(-5x + 5y = -20\).