2.11
Let \(< \) be the ordinary order on \( \mathbb{R} \). Prove that if \( 0 < a < b \) and \( 0 < c < d \), then \( ac < bd \). Also prove that if \( n = ab \), and all three are positive, then either \( a \leq \sqrt{n} \) or \( b \leq \sqrt{n} \).

Since \( c > 0 \) and \( a < b \), we can use O4 to say that \( ac < bc \). Since \( b > 0 \) and \( c < d \), by the same axiom we can write \( cb < db \). Now multiplication on the reals is commutative (M1), so that is the same as \( bc < bd \). But by transitivity (O3) that means \( ac < bd \).

For the second part, let’s assume the opposite - that both \( a \) and \( b > \sqrt{n} \). Then \( a > \sqrt{n} > 0 \) and \( b > \sqrt{n} > 0 \). By the first part, this means \( ab > (\sqrt{n})^2 \). But \((\sqrt{n})^2 = n \). Hence \( ab > n \), which is a contradiction. Thus our assumption wasn’t true; so at least one of \( a \) or \( b \) is less than or equal to \( \sqrt{n} \).

2.12(b,c)
Suppose we have an order satisfying axioms O1-O4. Prove, for \( a, b, x, y \) in the set, that if \( x > y \) then \( a-x < a-y \) and that if \( 0 < a < b \) then \( a^2 < b^2 \).

Assume \( x > y \). Then by Theorem 2.8, \(-x < -y \). By axiom O2, we can add \( a \); so \( (x) + a < (-y) + a \). Use commutativity (A1), and this is the same as \( a + (x) < a + (-y) \). By the definition of subtraction, we get what we wanted, \( a-x < a-y \).

Assume \( 0 < a < b \). Since \( b > 0 \) and \( a < b \), we can use O4 to write \( ab < b^2 \); similarly, since \( a > 0 \), we can write \( a^2 < ba \). Use A1 to rewrite this as \( a^2 < ab \); then transitivity (O3) gives that \( a^2 < b^2 \). We could also have used exercise 2.11 if it had been in the text.

2.14(a,b)
For a real number \( a \), the absolute value is defined as the largest number of the set \( \{a, -a\} \). Prove it can also be defined
\[
|a| = \begin{cases} 
  a & \text{for } a \geq 0 \\
  -a & \text{for } a < 0
\end{cases}.
\]

Also show that \( |a| = 0 \) only when \( a = 0 \).

To see they are the same definition, just check what happens for \( a \geq 0 \) and \( a < 0 \). If \( a > 0 \), then the set \( \{a, -a\} \) consists of the positive number \( a \) and the negative number \( -a \), of which \( a \) is larger; that is what the second definition predicts. Similarly, if \( a < 0 \), the set \( \{a, -a\} \) consists of the positive number \( -a \) and the positive number \( a \), of which \( -a \) is larger; that is what the second definition predicts. Finally, if \( a = 0 \), then the set \( \{a, -a\} \) is just the set \( \{0\} \), and the largest element of that set is 0, which is \( a \), as predicted by the second definition. So the definitions are the same. Note that the proof we just gave, if you check it carefully, says that for any nonzero number, its absolute value is positive (so not zero). Hence \( |a| = 0 \) only when \( a = 0 \).
2.15(a,b,d)
The set of odd natural numbers certainly has a least element; it is 1. The set of positive rational numbers does not; if you have a potential smallest number \( \frac{a}{b} \), note that this is greater than or equal to \( \frac{1}{b} \). But this is greater than \( \frac{1}{n+1} \), which is a smaller rational number. So you can always get a smaller positive rational number. As for the set of rational numbers greater than \( \pi \), this also does not have a least element. It is more tricky to see; the basic idea is that once you show that every positive number is greater than \( \frac{1}{n} \) for some (big) \( n \in \mathbb{N} \), taking a number \( x \) bigger than \( \pi \) leads us to look at \( x - \pi > \frac{1}{n} \) (for some \( n \)), and we use the same trick, but more carefully to ensure getting a rational number. You didn’t have to prove these last parts.

2.17(a,b,c)
Let \( S \) be a set with an addition satisfying A1-A4. Assume there is also an ordering \(<\) satisfying O1-O3 (note: not O4!). Two elements \( a \) and \( b \) are said to be consecutive if there is no element \( x \) in \( S \) satisfying \( a < x < b \).

If \( a \) and \( b \) are consecutive, what about \( a + c \) and \( b + c \)? Suppose there is an element \( x \) such that \( a + c < x < b + c \), so they are not consecutive. We are allowed (by O2) to add any number; let’s use \(-c\) (which exists by A4). Then \((a + c) + (-c) < x + (-c) < (b + c) + (-c)\) (use O2 twice). By associativity and the definition of subtraction, we get \( a + (c - c) < x - c < b + (c - c) \). By A4, this is \( a + 0 < x - c < b + 0 \), which is \( a < x - c < b \) by A3. But addition is a closed operation; hence \( a \) and \( b \) would be nonconsecutive. Since they’re consecutive, we have a contradiction, and \( a + c \) and \( b + c \) are consecutive.

Since 0 and 1 have already been shown to be consecutive (see Theorem 2.12), \( 0 + c < 1 + c \) for any \( c \); let \( c = n \). So \( n = 0 + n < 1 + n = n + 1 \). Thus by the first part, they are consecutive.

Finally, consider the set \( \mathbb{Q} \). Let \( a < b \) be two elements. There is an element between them. What is it? It’s \( \frac{a+b}{2} \), which is clearly still rational. Since \( 2a < a + b < 2b \), we divide by two (multiply by \( \frac{1}{2} \), using O4), we get \( a < \frac{a+b}{2} < b \). So no two elements are consecutive.

2.18
Read the problem from the text: if \(<\) were transitive, then since scissors < rock and rock < paper, we would expect scissors < paper. But in fact paper < scissors. So \(<\) is not transitive.