\textbf{HOMEWORK SOLUTIONS 01/03/00}

19.4

(ii) Calculate \( \int \frac{dx}{\sqrt{1+x^2}} \).

We use the substitution \( x = \tan u \). Hence \( dx = \sec^2 u \, du \), so as \( \sec^u = 1 + \tan^2 u \) the integral is \( \int \frac{\sec^2 u \, du}{\sqrt{1+\tan^2 u}} = \int \sec u \, du \). We know that a primitive of this is \( \log(\sec u + \tan u) \), so by substitution again it is \( \log(\sec(\arctan x) + x) \). Finally, we recall from a previous homework that \( \cos \arctan x = \frac{1}{\sqrt{1+x^2}} \), so we invert for \( \sec \) and the integral may be expressed as \( \log(\sqrt{1+x^2} + x) \).

(vii) Calculate \( \int \sqrt{1-x^2} \, dx \).

We use the substitution \( x = \sin u \). Then \( dx = \cos u \, du \), so since \( \cos^2 u + \sin^2 u = 1 \), the integral transforms into \( \int \cos^2 u \, du \). Since \( \cos 2u = \cos^2 u - \sin^2 u \), by adding \( 1 = \cos^2 u + \sin^2 u \) to both sides and dividing by two we see that \( \cos^2 u = \frac{1+\cos 2u}{2} \); thus the integral simplifies to \( \int \frac{1}{2} du + \int \frac{\cos 2u}{2} du = \frac{u}{2} + \frac{\sin 2u}{4} = \frac{u + \sin u \cos u}{2} \). We then put back \( u = \arcsin x \), to get \( \frac{\arcsin x}{2} + \frac{x \sqrt{1-x^2}}{2} \), recalling again a previous homework.

19.6

(ii) Calculate \( \int \frac{2x+1}{x^2-3x+3} \, dx \).

First note that the denominator is \( (x-1)^2 \). One may do this one several ways; the slickest I saw was to write \( 2x+1 \) as \( 2(x-1)+3 \). Then, since the substitution \( u = x-1 \) gives \( du = dx \), a very simple computation and a little cancellation of \( u \) shows that the integral is \( \int \frac{2}{x-1} + \frac{3}{2(x-1)^2} \). This is clearly \( -2u^{-1} - 3/2u^{-2} \), or
\[
-\frac{2}{x-1} - \frac{3}{2(x-1)^2}.
\]

19.8

(ii) Calculate \( \int \frac{\sec^2 x}{\sin^2 x} \, dx \).

Again, there are multiple ways to solve this one. My own favorite is to note that the integrand may also be written \( \csc^2 x + \frac{\sec^2 x}{\sin^2 x} \), so without further ado (save a simple substitution) we find the answer is \( -\cot x - \csc x \).

19.11

(i) Calculate \( \int \frac{dx}{\csc x} \) using the world’s sneakiest substitution.

We are told that we wish to use \( x = 2 \arctan t \), \( dx = \frac{2}{1+t^2} \, dt \). This gives us (we are also told) \( \sin x = \frac{2t}{\sqrt{1+t^2}} \). So the integral becomes
\[
\int \frac{\frac{2}{1+t^2} \, dt}{1 + \frac{4t^2}{1+t^2}}
\]
which after some simplification yields \( \int \frac{2t}{1+2t+4t^2} \). But as the denominator is just \( (1 + t)^2 \), we can integrate and recover \( \frac{2}{1+t} \), which then resubstitutes to get \( \frac{1}{1+\tan(x/2)} \).

Finally, we compare the usual method for such a problem, that is rewriting the integrand as \( \frac{1-\sin x}{1-\sin^2 x} = \frac{1-\sin x}{\cos^2 x} \), which we integrate as in problem 8 to see that
\[-\frac{2}{1 + \tan(x/2)} = \tan x - \sec x + C\] for a constant $C$ which by letting $x = 0$ is easily seen to be $C = -1$.

19.14
Suppose that $f''$ is continuous and that $\int_0^\pi [f(x) + f''(x)] \sin x \, dx = 2$. Given that $f(\pi) = 1$, compute $f(0)$.

We know that $\int_0^\pi [f(x) + f''(x)] \sin x \, dx = \int_0^\pi f(x) \sin x \, dx + \int_0^\pi f''(x) \sin x \, dx$.

We consider each of these integrals separately for the moment, integrating each by parts; the first one will have $u_1 = f(x)$ and $dv_1 = \sin x \, dx$, while the second will have $u_2 = \sin x$ and $dv_2 = f''(x) \, dx$. We integrate by parts and expand and get

\[
f(\pi)(-\cos \pi) - f(0)(-\cos 0) - \int_0^\pi f'(x) \cos x \, dx
\]
\[
+ f'(\pi) \sin \pi - f'(0) \sin 0 - \int_0^\pi f''(x) \cos x \, dx.
\]

Since $\sin \pi = \sin 0 = 0$ and the integrals cancel out, we are left with $2 = f(\pi)(-\cos \pi) - f(0)(-\cos 0) = f(\pi) + f(0)$. Since $f(\pi) = 1$, $f(0) = 1$ as well.

19.15
(a) Find $\int \arcsin x \, dx$.

We use integration by parts. In particular let $u = \arcsin x$ and $dv = 1$; then $\int \arcsin x \, dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} \, dx$, which by using the substitution $y = 1-x^2$ easily reduces to $x \arcsin x + \sqrt{1-x^2}$.

19.18
Express $\int x^2 e^{-x^2} \, dx$ in terms of $\int e^{-x^2} \, dx$.

We use integration by parts. Let $u = x$, $dv = xe^{-x^2}$. A short calculation reveals that $v = -e^{-x^2}/2$. Then by parts we quickly see $\int x^2 e^{-x^2} \, dx = -xe^{-x^2}/2 + \frac{1}{2} \int x^2 e^{-x^2} \, dx$.

19.19
Prove that the function $f(x) = e^x/(e^{5x} + e^x + 1)$ has an elementary primitive.

Let’s consider the substitution $u = e^x$, $du = e^x \, dx$. Then $\int f(x) \, dx = \int 1/(u^5 + u + 1) \, du$. Since the latter integrand is a rational function, by Spivak’s comment on page 376, the latter integral has an elementary primitive, and so does $f(x)$. 