20.1

Find the Taylor polynomials for the following functions of the indicated degree and at the indicated points:

(ii) \( f(x) = e^{\sin x} \) degree 3, at 0.

First we calculate the appropriate derivatives: \( f'(x) = \cos xe^{\sin x}; \quad f''(x) = \cos^2 xe^{\sin x} - \sin xe^{\sin x}; \quad \) and \( f'''(x) = -3\cos x \sin xe^{\sin x} + \cos^3 xe^{\sin x} - \cos xe^{\sin x}. \)

Then the Taylor polynomial is \( P_{3,0}(x) = f(0) + f'(0)x + (f''(0)/2)x^2 + (f'''(0)/6)x^3, \)

which after plugging in the various values is \( 1 + x + \frac{x^2}{2}. \)

(viii) \( f(x) = x^3 + x^2 + x; \) degree 4, at 1.

Again, we calculate derivatives: \( f'(x) = 5x^4 + 3x^2 + 1; \quad f''(x) = 20x^3 + 6x; \quad f'''(x) = 60x^2 + 6 \); and \( f^{[4]}(x) = 120x. \) The T.P. is \( P_{4,1}(x) = f(1) + f'(1)(x - 1) + (f''(1)/2)(x - 1)^2 + (f'''(1)/6)(x - 1)^3 + (f^{[4]}(1)/24)(x - 1)^4. \) That is, after substitution, \( 3 + 9(x - 1) + 13(x - 1)^2 + 11(x - 1)^3 + 5(x - 1)^4. \)

(x) \( f(x) = \frac{1}{1 + x^2}; \) degree \( n, \) at 0.

Here, since we know that for any \( k, \ (y^{(n)}(x))^k = n(n - 1) \cdots (n - (k - 1)) y^{n-k} \)

(\( \text{if } n > 0 \) but \( n - k \leq 0, \) then of course this gives zero), then by the chain rule we see that in fact \( f^{[k]}(x) = (-1)^k k! (1 + x)^{-k}. \) Then we see that \( f^{[k]}(0) = (-1)^k k!. \) So the T.P. \( f(0) + (f'(0)/1!)(x) + (f''(0)/2!)(x)^2 + \cdots + (f^{[n]}(0)/n!)(x)^n \)

is simply \( 1 - x + x^2 - \cdots + (-1)^n x^n. \)

20.2

Write each of the following polynomials in \( x \) as a polynomial in \( (x - 3). \)

It is only necessary to compute the Taylor polynomial at 3 of the original polynomial in the same degree, as the corollary to Theorem 3 makes clear.

(ii) \( x^4 - 12x^3 + 44x^2 + 2x + 1 \)

We compute the first four derivatives: \( 4x^3 - 36x^2 + 88x + 2; \quad 12x^2 - 72x + 88; \quad 24x - 72; \quad 24 \).

If we evaluate at \( x = 3 \) and divide out by the appropriate factorials, then plug into the T.P. formula, we get \( P_{4,3}(x) = 160 + 50(x - 3) - 10(x - 3)^2 + (x - 3)^4. \)

(iv) \( ax^2 + bx + c \)

The first two derivatives are \( 2ax + b \) and \( 2a. \) We evaluate at \( x = 3 \) and remember to include \( 2!, \) yielding \( P_{2,3}(x) = (9a + 3b + c) + (6a + b)(x - 3) + a(x - 3)^2. \)

20.7

Show that the Taylor polynomial of degree \( n \) for \( f(x) = (1 + x)^\alpha \) at \( 0 \) is \( P_{n,0}(x) = \sum_{k=0}^{n} \binom{\alpha}{k} x^k. \)

Note: The problem also asks for the Lagrange and Cauchy forms. If you do not understand how the first expression happens, simply plug in what you get for the derivatives. The remaining expressions are elementary manipulations with binomial coefficients, and if you can’t do them, then go back to chapter two and try to figure out how they work.

We see that \( (1 + x)^\alpha = \alpha(1 + x)^{\alpha-1}, \) and if \( (1 + x)^\alpha\binom{\alpha}{k} = \alpha(\alpha - 1) \cdots (\alpha - (k - 1))(1 + x)^{\alpha-k}, \) then of course \( (1 + x)^\alpha\binom{\alpha}{k+1} = \alpha(\alpha - 1) \cdots (\alpha - k)(1 + x)^{\alpha-(k+1)}, \)

so by induction we have the result for all \( k. \) Then recall that \( \alpha(\alpha - 1) \cdots (\alpha - (k -
1))/k! = \binom{a}{k}. Thus, as (1 + 0)^\beta = 1 for all \beta, we plug in these coefficients and see 
\[ P_{n,0}(x) = \binom{a}{0} + \binom{a}{1} x + \cdots + \binom{a}{n} x^n = \sum_{k=0}^{n} \binom{a}{k} x^k \], as desired.

20.8

(ii) Suppose that \( a_i \) and \( b_i \) are the coefficients in the Taylor polynomials at \( a \) of \( f \) and \( g \) respectively. In other words, \( a_i = f^{(i)}(a)/i! \) and \( b_i = g^{(i)}(a)/i! \). Find the coefficients \( c_i \) of the function \( fg \) in terms of the \( a_i \)'s and \( b_i \)'s.

First we note that \( c_0 = \text{simply } f(0)g(0) = a_0b_0 \). We also observe that \( i!a_i = f^{(i)}(a) \) and \( i!b_i = g^{(i)}(a) \).

Next, we want to see what \( (fg)^{(k)} \) is in terms of derivatives of \( f \) and \( g \). Since 
\[ (fg)' = fg' + f'g \] and \( (fg)^{(n)} = fg^{(n)} + 2f'g' + f''g \), we suspect the general formula might be \( (fg)^{(k)} = \sum_{i=0}^{k} \binom{k}{i} f^{(i)}(a) g^{(k-i)}(a) \). Clearly it is true for \( k = 0, 1, 2 \). We assume true for \( n \). Then by the induction assumption \( (fg)^{(n+1)} = [(fg)^{(n)}]' \), or \( \sum_{i=0}^{n} \binom{n}{i} f^{(i)}(a) g^{(n-i)}(a) \). We then have the sum of two sums, \( \sum_{i=0}^{n} \binom{n}{i} f^{(i)}(a) g^{(n-i)}(a) + \sum_{i=0}^{n} \binom{n}{i} f^{(i)}(a) g^{(n-i)}(a) \), by the product rule. But we can consider the first sum as \( \sum_{i=0}^{n} \binom{n}{i} f^{(i)}(a) g^{(n-i)}(a) \). Then behold, we can combine the two sums into \( \binom{n}{0} f^{(0)}(a) g^{(n)}(a) \) and \( n - (i - 1) = (n+1) - i \), so by the summing property of binomial coefficients this is the sum \( \sum_{i=0}^{n} \binom{n+1}{i} f^{(i)}(a) g^{(n+1-i)}(a) \). Hence by induction we are done with this step.

Now we simply write the derivatives in terms of the \( a_i \)'s and \( b_i \)'s. Then we have \( (fg)^{(k)}(a) = \sum_{i=0}^{k} \binom{k}{i} a_i (k-i)! b_i \). But \( \binom{k}{i} = \frac{k!}{i!(k-i)!} \), so in fact this sum is \( \sum_{i=0}^{k} \frac{k!}{i!(k-i)!} a_i (k-i)! b_i = \sum_{i=0}^{k} k! a_i b_i \). Finally, we recall that the Taylor polynomial coefficient is \( (fg)^{(k)}(a)/k! \). That is, \( c_i = \sum_{i=0}^{k} a_i b_i \).

20.9

(a) Prove that the Taylor polynomial of \( f(x) = \sin(x^2) \) of degree \( 4n + 2 \) at 0 is 
\[ x^2 - \binom{2}{3} x^4 + \binom{2}{5} x^6 - \cdots + (-1)^n \binom{2n}{2n+2} \frac{x^{4n+2}}{(2n+1)!} \].

In general, consider a function \( g \) such that \( \lim_{x \to 0} g(x) = 0 \). I claim that also \( \lim_{x \to 0} g(x^2) = 0 \). Certainly \( \lim_{x \to 0} g(x^2) = 0 \), since this is just a specific instance of the first limit. But then for an \( \epsilon > 0 \), we have a \( \delta > 0 \) such that \( |x| < \delta \) implies \( |g(x)| < \epsilon \). But then for any \( \epsilon > 0 \), we choose \( \delta' = \sqrt{\delta} \); that implies that for \( |x| < \delta', |x^2| < \delta \) so that \( |g(x^2)| < \epsilon \). So we have found a \( \delta' \) for any \( \epsilon \), so indeed \( \lim_{x \to 0} g(x^2) = 0 \).

Now we note that the polynomial in question is simply \( P_{2n+1,0}(x) \) for \( sin \), but evaluated at \( x^2 \). We know that \( \lim_{x \to 0} \frac{\sin x - P(x)}{x^{4n+2}} = 0 \) by Theorem 1; however, by the above result we also know that \( \lim_{x \to 0} \frac{x^2 - P'(x)}{x^{4n+2}} = 0 \), so by the uniqueness of the T.P. which Lebovitz proved in class, we are done.

20.15

Suppose that \( f \) is twice differentiable on \((0, \infty)\) and that \( |f(x)| \leq M_0 \) for all \( x \geq 0 \), while \( |f''(x)| \leq M_2 \) for all \( x > 0 \). Prove that for all \( x > 0 \) we have \( |f'(x)| \leq \frac{2}{x} M_0 + \frac{1}{2} M_2 \) for all \( h > 0 \). Indeed, show that we have \( |f'(x)| \leq 2 \sqrt{M_0 M_2} \).

We consider the Taylor polynomial \( f(x) = f(a) + f'(a)(x-a) + \frac{f''(t)}{2}(x-a)^2 \) (true for some \( t \) between \( x \) and \( a \)) as the Lagrange form of the remainder. We separate \( f'(a) \); \( f'(a)(x-a) = f(x) - f(a) - \frac{f''(t)}{2}(x-a)^2 \). But then we divide
by \((x - a)\) (so we assume \(x \neq a\)) and take absolute values; this yields \(|f'(a)| = \left| \frac{f(x) - f(a)}{x - a} - \frac{f''(t)}{2}(x - a) \right|\). By the triangle inequality applied a few times, this is \(|f'(a)| \leq \frac{|f(x)| + |f(a)|}{|x - a|} + \frac{|f''(t)|}{2}|x - a|\). By applying the bounds given in the statement of the problem and letting \(|x - a| = h (h > 0\) as we assume \(x \neq a\)), we arrive at \(|f'(a)| \leq \frac{2}{h} M_0 + \frac{M}{2} H_2\) as desired (since we can always find some \(x \neq a\) but \(x > 0\) for \(h > 0\)).

But is there a minimum for this sort of bound? Consider as a function of \(h\)
\(g(h) = \frac{2}{h} M_0 + \frac{M}{2} H_2\). Then \(g'(h) = -\frac{2}{h^2} M_0 + \frac{M}{2} h^2\). Letting this equal zero, we see that \(0 = -\frac{2}{h^2} M_0 + \frac{M}{2} h^2\) implies \(M h^2 = \frac{2}{h^2} M_0\), or \(h^2 = \frac{4 M_0}{M h^2}\), which yields \(h = 2 \frac{M h^2}{M_0}\). And this is indeed a minimum and not just a critical point, because \(g''(h) = \frac{2}{h^3} M_0\), which is always positive. Then we plug in the value of \(h\) for that critical point into the formula \(|f'(a)| \leq \frac{2}{h} M_0 + \frac{M}{2} H_2\) to get \(|f'(a)| \leq \frac{2}{\sqrt{2} M_0} M_0 + \frac{2\sqrt{2} M_2}{4 M_0} M_2 = \sqrt{M_0 M_2} + \sqrt{M_0 M_2},\) or \(|f'(x)| \leq 2 \sqrt{M_0 M_2},\) which is what we wanted.

20.16
(a) Prove that if \(f''(a)\) exists, then
\[ f''(a) = \lim_{h \to 0} \frac{f(a + h) + f(a - h) - 2f(a)}{h^2}. \]

We know that \(\lim_{x \to a} \frac{f(x) - f(a)}{(x - a)^2} = 0\), where \(P_{2,a}\) is the Taylor polynomial for \(f\) of degree 2 at \(a\). We may of course write \(x = a + h\) or \(x = a - h\) if we so desire, giving us \(\lim_{h \to 0} \frac{P_{2,a}(a + h) - P_{2,a}(a)}{h^2} = 0\) and \(\lim_{h \to 0} \frac{P_{2,a}(a - h) - P_{2,a}(a - h)}{h^2} = 0\). We add these equations and substitute in the actual Taylor polynomials. Then
\[ \lim_{h \to 0} \frac{f(a + h) + f(a - h) - (f(a) + f'(a) h + f''(a) h^2/2) - (f(a) + f'(a) (-h) + f''(a) h^2/2)}{h^2} = 0; \]
then we simplify the expression and note that \(f''(a)\) is independent of \(h\), so we see that
\[ f''(a) = \lim_{h \to 0} \frac{f(a + h) + f(a - h) - 2f(a)}{h^2}. \]

(b) Let \(f(x) = x^2\) for \(x \geq 0\), and \(-x^2\) for \(x \leq 0\). Show that \(\lim_{h \to 0} \frac{f(0 + h) + f(0 - h) - 2f(0)}{h^2}\) exists, even though \(f''(0)\) does not.

We simply plug in the values of the functions. First, \(f(0) = 0\). Next, \(f(h) = \frac{h^2}{2} - h^2\), which is \(1/2\) for \(h > 0\) or \(h < 0\). Finally, this means \(\lim_{h \to 0} \frac{f(0 + h) + f(0 - h) - 2f(0)}{h^2} = \lim_{h \to 0} \frac{\frac{h^2}{2} - h^2}{h^2} = \lim_{h \to 0} \frac{h^2}{2} - \lim_{h \to 0} h^2 = 0,\)