 HOMEWORK SOLUTIONS 16/02/00

4(App).5
Consider a hyperbola, where the difference of the distances from a point to each of the foci is the constant $2a$, and choose one focus at $O$ and the other at $(-2ea, 0)$. Show that we obtain the exact same equation in polar coordinates as we obtained for the ellipse, $r = \frac{\Lambda}{1 + \epsilon \cos \theta}$.

First note that if we look at a point $(x, y)$ on the hyperbola and the line through $(x, y)$ and the origin, the angle subtended by this line and $x$-axis is a unique $\theta$ for $0 \leq \theta < 2\pi$. This can be noted by examining the graph or more rigorously by solving the equations we will consider and noting unique solutions.

For $(x, y)$ on the curve, then, the distance to the origin will be denoted $r$, so that by Pythagoras $r^2 = x^2 + y^2$. By definition of the hyperbola, the distance to the other focus $f$ is then $2a + r$, at least for $r < \text{dist}(f, (x, y))$; for the other case the distance is $r - 2a$. Using Pythagoras again gives us $(2a + r)^2 = (2ea + x)^2 + y^2$, which can be expanded to

$$4a^2 + 4ar + r^2 = x^2 + y^2 + 4e^2a^2 + 4eax.$$

We may subtract $r^2 = x^2 + y^2$ from this and divide both sides by $4a$; then we are left with $a + r = ae^2 + ex$. Letting $\Lambda = a(e^2 - 1)$ as before and recalling that $x = r \cos \theta$, this reduces to $r = \Lambda + er \cos \theta$. If we use the other distance then we get $r = \Lambda - er \cos \theta$; this latter designation yields $r = \frac{\Lambda}{1 + \epsilon \cos \theta}$ as desired.

4(App).6
Consider the set of points $(x, y)$ such that the distance $(x, y)$ to $O$ is equal to the distance from $(x, y)$ to the line $x = a$. Show that the distance to the line is $a - r \cos \theta$ and conclude that the equation can be written $a = r(1 + \cos \theta)$.

Certainly the distance to the line $x = a$ is simply $a - x$, since we take the shortest distance and that would be the distance in the $x$-axis direction. But if we consider the angle from the $x$-axis to the line through the origin and $(x, y)$ to be $\theta$, then $x = r \cos \theta$, where $r$ is the distance from the origin to $(x, y)$; then we have that the distance to the line $x = a$ is $a - r \cos \theta$. Then by the definition given for the parabola, $r = a - r(\cos \theta)$, or $a = r(1 + \cos \theta)$.

4(App).7
Now, for any $\Lambda$ and $\epsilon$, consider the graph in polar coordinates of the equation $r = \frac{\Lambda}{1 + \epsilon \cos \theta}$, which implies $r = \Lambda - \epsilon x$. Show that the points satisfying this equation satisfy

$$(1 - \epsilon^2)x^2 + y^2 = \Lambda^2 - 2\Lambda \epsilon x.$$

Using Problem 4.16, show that this is an ellipse for $\epsilon < 1$, a parabola for $\epsilon = 1$, and a hyperbola for $\epsilon > 1$.

A slick solution several people came up with follows. We have the equation given, $r = \Lambda - \epsilon x$. Certainly any point satisfying this will also satisfy the same expression but with both sides squared. So we have $r^2 = \Lambda^2 - 2\Lambda \epsilon x + \epsilon^2 x^2$. But we
know, again by Pythagoras, that \( x^2 + y^2 = r^2 \). Then we have, by expanding and combining the \( x^2 \) terms, the first result. So the points also satisfy this equation.

Problem 4-16 claims that a set of all points satisfying some equation \( Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \) is a parabola, ellipse, or hyperbola. Indeed, if \( A = 0 \), then we complete the square for \( y \) and put \( x \) on the other side and get a classical equation for the parabola; above if \( \epsilon = 1 \) we have this phenomenon. If \( A \) and \( B \) are both positive, then we may complete the square for both \( x \) and \( y \) (and divide out by everything necessary to remove coefficients), which yields an equation of the form \( (x - \alpha)^2/G^2 + (y - \beta)^2/H^2 = 1 \), which is the equation for an ellipse. If \( \epsilon < 1 \) in the above expression we get this. Finally, if \( \epsilon > 1 \) then we have a formula equivalent to \( A < 0 \) but \( C > 0 \), so if we complete the square we will have to consider the negative coefficient, i.e. we get something of the form \( -(x - \alpha)^2/G^2 + (y - \beta)^2/H^2 = 1 \), which is the expression for a hyperbola.

Problem from Lebowitz

Express the semi-axes \( a \) and \( b \) of an ellipse in terms of the parameters \( \Lambda \) and \( \epsilon \).

We see from the chapter in question that \( a \) and \( b \) are precisely the constants appearing in the equation \( \frac{x^2}{\Lambda} + \frac{y^2}{\epsilon} = 1 \). This makes sense since certainly, for such an ellipse centered at the origin, \( |y| \leq b \) and \( |x| \leq a \) or else we would need complex coordinates! To determine the points of the foci \((\pm \epsilon, 0)\) we then note that by considering the point \((0, b)\) we have (again by Pythagoras) that \( 2c^2 + 2b^2 = 2a^2 \) (by a nice computation from chapter 4, that the sum of the distances to the foci is \( 2a \) for such an ellipse, and the symmetry), or \( c^2 = a^2 - b^2 \). In this case we see that by definition \( \epsilon = c/a \). So since the \( a \)s are the same, since we define \( \Lambda = (1 - \epsilon^2)a \), we have \( a = \frac{\Lambda}{\sqrt{1 - \epsilon^2}} \). By \( \epsilon = c/a \), we further see \( b = \sqrt{a^2 - c^2} = \sqrt{a^2 - \epsilon^2 a^2} = a\sqrt{1 - \epsilon^2} = \frac{\Lambda}{\sqrt{1 - \epsilon^2}} \), just as Spivak presents it.

12(App).1

(a) For a function \( f \), the ‘point-slope form’ of the tangent line at \((a, f(a))\) can be written as \( y - f(a) = (x - a)f'(a) \), so that the tangent line consists of all points of the form \((x, f(a) + (x - a)f'(a))\). Conclude that the tangent line consists of all points of the form \((a + s, f(a) + sf'(a))\).

(b) If \( c \) is the curve \( c(t) = (t, f(t)) \), conclude that the tangent line of \( c \) at \((a, f(a))\) is the same as the tangent line of \( f \) at \((a, f(a))\).

Assuming the first part, we consider part (b). From the text, we define the tangent line of \( c \) at \( c(a) = (a, f(a)) \) to be the the set of all points \( c(a) + sc'(a) \), where \( s \) ranges along all the reals. But also from the text we have \( c'(t) = (1, f'(t)) \); hence the tangent line at \( c(a) \) is the set of points \( c(a) + sc'(a) = (a, f(a)) + s(a, f'(a)) = (a + s, f(a) + sf'(a)) \). So by part (a), this is the same as the usual definition of the tangent line. As for the first part itself, one may consider that any real number \( x \) may be uniquely written as \( x = a + s \), for some \( s \), given that \( a \) is fixed, as it is here. Then this is a simple substitution.

12(App).3

Suppose that \( x = u(t), y = v(t) \) is a parametric representation of a curve, and that \( u' \neq 0 \) on some interval.

(a) Show that on this interval the curve lies along the graph of \( f = v(u^{-1}) \). Since apparently we are considering a continuous derivative (otherwise I see no reason for this to be true), then \( u' \neq 0 \) implies that \( u' > 0 \) or \( u' < 0 \) on the entire interval; that is, \( u \) is itself decreasing or increasing on the whole interval. Then \( u \) is one-to-one and
so $u^{-1}$ exists. Then any point $(u(t), v(t)) = (u(t), v(u^{-1}(u(t)))) = (u(t), f(u(t)))$, so since that is on the graph of $f$ for $x = u(t)$, we are done.

(b) Show that at the point $x = u(t)$ we have $f'(x) = \frac{v'(t)}{u'(t)}$. But note that the second expression is more formally

$$\lim_{g \to 0} \frac{v(t + g) - v(t)}{g} = \lim_{g \to 0} \frac{u(t + g) - u(g)}{u(t + g) - u(g)} = \lim_{g \to 0} \frac{v(t + g) - v(t)}{u(t + g) - u(g)}$$

since $u' \neq 0$ implies that close to $g = 0 u(t + g) - u(g) \neq 0$. But we have that $x = u(t)$ so $t = u^{-1}(x)$; further, since $u^{-1}$ is continuous, for every $g$ there is an $h$ such that $u^{-1}(x + h) = t + g$, in a continuous way (i.e. $h$ goes to zero as $g$ does).

Thus we have

$$\lim_{g \to 0} \frac{v(t + g) - v(t)}{u(t + g) - u(g)} = \lim_{g \to 0} \frac{v(t + g) - v(t)}{u^{-1}(x + h) - u^{-1}(x)} = \lim_{g \to 0} \frac{v(u^{-1}(x + h)) - v(u^{-1}(x))}{h(g)}$$

which is just the definition of $f'$, by the comment on $h$ above.

13.16

Prove that $\int_a^b f(x)dx = c \int_a^b f(cx)dx$. Since we have not yet defined integrals $\int_a^b f$ for $x > y$, I will assume that $c > 0$ (for $c = 0$ it is vacuously true).

We proceed similarly as in Problem 14. Consider the second integral. This exists only if

$$\sup\{L(f(cx), P) | P \text{ a partition of } [a, b]\} = \inf\{U(f(cx), P) | P \text{ a partition of } [a, b]\}$$

where $U$ and $L$ are the usual upper and lower sums. Note that $c > 0$ and $x > y$ implies $cx > cy$, irrespective of whether $x$ and $y$ are positive or negative. Thus, for a typical $P$, $a = t_0 < t_1 < \cdots < t_n = b$, we get a partition $Q$ of $[ca, cb]$, namely $a = ct_0 = s_0 < ct_1 = s_1 < \cdots < ct_n = s_n = cb$. The other implication takes some reasoning; if we have $x < y$, then since $c > 0$, we can reverse the process to get a partition for $[a, b]$ from a partition of $[ca, cb]$. Just as in the problem above, the processes are clearly inverse to each other.

Further, note that $M_i = \sup\{f(cx) | t_{i-1} < x < t_i\} = \sup\{f(x) | ct_{i-1} < x < ct_i = s_i = t_i\} = N_i$, and similarly for inf of the same quantities. But here we also need to take into account that $N_i(s_i - s_{i-1}) = N_i(ct_i - ct_{i-1}) = cM_i(t_i - t_{i-1})$. That is, using all this information, $cU(f(cx), P) = U(f(x), Q)$ and likewise $cL(f(cx), P) = L(f(x), Q)$. Now we can use the one-to-one correspondence outlined above between partitions of $[a, b]$ and $[ca, cb]$; from these equations we find that $\sup cL(f(cx), P) = c \sup L(f(cx), P) = \sup L(f(x), Q)$ and inf $cU(f(cx), P) = c \inf U(f(x), Q)$. But since $f$ is integrable over $[ca, cb]$ the third and sixth terms are equal, so the second and fifth are equal too. Thus by definition $\int_a^b f(cx)dx$ exists, and

$$\int_a^b f(x)dx = c \int_a^b f(cx)dx$$

holds true.

13.17

Given that the area enclosed by the unit circle $x^2 + y^2 = 1$ is $\pi$, show that the area of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is $\pi ab$.

First we note that this ellipse is centered at the origin, and indeed is symmetric about the $x$-axis, since $(-x)^2 = x^2$. We note that for a given $x$, there are
precisely the two values \( y = \pm \sqrt{\frac{b^2(1 - x^2/a^2)}{a^2}} \) which determine a point on the ellipse. Taken together, these things imply we can consider the area under the curve \( y = \sqrt{\frac{b^2(1 - x^2/a^2)}{a^2}} \), just as in the case for the circle; note that this is only defined for \(-a \leq x \leq a\). That is, we compute (and double) the integral
\[
b \int_{-a}^{a} \sqrt{1 - \frac{x^2}{a^2}} \, dx.
\]
But \( x^2/a^2 = (x/a)^2 \), so by using problem 16 (for the constant \( c = 1/a \)) we have that this is the same as
\[
ba \int_{-1}^{1} \sqrt{1 - x^2} \, dx.
\]
By the definition of \( \pi \), this is merely \( ab \pi /2 \). So the area of the ellipse is \( \pi ab \).