13.5

Evaluate without doing any computations:
(i) \( \int_{-1}^{1} x^3 \sqrt{1-x^2} \, dx \).
(ii) \( \int_{-1}^{1} (x^5 + 3) \sqrt{1-x^2} \, dx \).

We can easily evaluate both without computation. The first function has \(-f(x) = f(-x)\), so over the indicated interval the integral is simply 0. The second function, upon inspection, can be broken up into two parts by the distributive axiom and Theorem 5. The first is similar to the first integral, except now the power is 5 instead of 3, which does not change the fact that it has zero integral. The second half is simply the formula for computing three times the area under a half-circle, which when it has radius 1 is simply \( \frac{3\pi}{2} \).

13.7

Decide which of the following functions are integrable on \([0, 2]\), and calculate the integral when you can.
(ii) \( f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ x - 2, & 1 < x \leq 2 \end{cases} \).

Since this function is continuous except at one point, by our theorem from class and the theorem that a continuous function is integrable, \( f \) is integrable. Further, one can see by Th. 4 that this is the sum of the integral of \( x \) on the first interval and the integral of \( x - 2 \) on the second. Then by Th. 5 it is really the sum

\[
\int_{0}^{1} x \, dx + \int_{1}^{2} \frac{x}{2} \, dx = \frac{3}{2}.
\]

which by Th. 4 again and results from the chapter is \( \frac{3}{2} - 2 = 0 \).

(iv) \( f(x) = \begin{cases} x + [x], & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases} \).

One can see that, just as in the example on page 255 of Spivak, on any partition \( P = \{t_0, \ldots , t_n\} \), any \( m_i = 0 \) and most \( M_i > 0 \), so the upper and lower sums are not equal. Hence this is not integrable.

13.8

Find the areas of the regions bounded by
(i) the graphs of \( f(x) = x^2 \) and \( g(x) = -x^2 \) and the vertical lines through \((-1,0)\) and \((1,0)\).

Now this is clearly twice the area bounded by \( f \), the lines, and the \( x \)-axis, hence four times the area bounded by \( f \), the \( y \)-axis, the \( x \)-axis, and the second line (by symmetry). Then by the formula Spivak derives for \( \int_{0}^{1} x^2 \, dx \) and our presentation of area, we see that the full area is \( 4 \left( \frac{1}{3} \right) = \frac{4}{3} \).
(vi) the graph of $f(x) = \sqrt{x}$, the horizontal axis, and the vertical line through $(2, 0)$.

At first this seems intractable without the fundamental theorem of calculus (coming soon!). But this is an increasing function (it’s the next chapter, so check it yourself now), so we can define the inverse function, which is of course $g(x) = x^2$ (check this by composition of $f$ and $g$). Since $f(2) = \sqrt{2}$, we know that this area is the same as the area of the region between the vertical axis, the graph of $g(x) = x^2$, and the horizontal line through $(0, 2)$. This is the area of the rectangle of height 2 and width $\sqrt{2}$ minus the area under the graph of $g$ from 0 to $\sqrt{2}$. So finally we can compute the area of our original region; it is $2\sqrt{2} - \int_0^{\sqrt{2}} x^2 \, dx = \frac{4\sqrt{2}}{3}$.

13.12

If $a < b < c < d$ and $f$ is integrable on $[a, d]$, prove that $f$ is integrable on $[b, c]$.

When Spivak says don’t work hard, he means it. By Theorem 4, $f$ is integrable on $[a, c]$ as $a < c < d$. Again for this reason $f$ is integrable on $[b, c]$, since $a < b < c$.

13.14

Prove that $\int_a^b f(x) \, dx = \int_a^{b+c} f(x-c) \, dx$.

Consider the second integral. This exists only if

$$\sup\{L(f(x-c), P) \mid P \text{ a partition of } [a + c, b + c]\} = $$

$$\inf\{U(f(x-c), P) \mid P \text{ a partition of } [a + c, b + c]\}$$

where $U$ and $L$ are the usual upper and lower sums. But now note that for a typical $P$, $a + c = t_0 < t_1 < \cdots < t_n = b + c$, we get a partition $Q$ of $[a, b]$, namely $a = s_0 < t_1 = s_1 < \cdots < t_n = s_n = b$ (and vice versa). Further, note that $\sup\{f(x) \mid s_{i-1} < x < s_i\} = \sup\{f(x-c) \mid s_{i-1} + c = t_{i-1} < x < t_i = s_i + c\}$, and similarly for $\inf$ of the same quantities. That means (since the intervals in the partitions do not change length) that $U(f(x-c), P) = U(f(x), Q)$, and similarly $L(f(x-c), P) = L(f(x), Q)$. But since we have the one-to-one correspondence outlined above between partitions of $[a + c, b + c]$ and $[a, b]$, from these equations we find that $\sup L(f(x-c), P) = \sup L(f(x), Q)$ and $\inf U(f(x-c), P) = \inf U(f(x), Q)$. But since $f$ is integrable over $[a, b]$ the second and fourth terms are equal, so the first and third are equal too. Thus by definition $\int_a^{b+c} f(x-c) \, dx$ exists and is equal to $\int_a^b f(x) \, dx$.

13.16

Prove that $\int_{ca}^b f(x) \, dx = c \int_a^b f(cx) \, dx$.

Since we have not yet defined integrals $\int_x^y f$ for $x > y$, I will assume that $c > 0$ (for $c = 0$ it is vacuously true).

We proceed similarly as in Problem 14. Consider the second integral. This exists only if

$$\sup\{L(f(cx), P) \mid P \text{ a partition of } [a, b]\} = \inf\{U(f(cx), P) \mid P \text{ a partition of } [a, b]\}$$

where $U$ and $L$ are the usual upper and lower sums. Note that $c > 0$ and $x > y$ implies $cx > cy$, irrespective of whether $x$ and $y$ are positive or negative. Thus, for a typical $P$, $a = t_0 < t_1 < \cdots < t_n = b$ we get a partition $Q$ of $[ca, c]$, namely $ca = ct_0 = s_0 < ct_1 = s_1 < \cdots < ct_n = s_n = cb$. The other implication takes some reasoning; if we have $x < y$, then since $c > 0$, $\frac{x}{c} < \frac{y}{c}$ and we can reverse the process
to get a partition for $[a, b]$ from a partition of $[ca, cb]$. Just as in the problem above, the processes are clearly inverse to each other.

Further, note that $M_i = \sup\{f(x) \mid t_{i-1} < x < t_i\} = \sup\{f(x) \mid ct_{i-1} = s_{i-1} < x < s_i = t_i\} = N_i$, and similarly for inf of the same quantities. But here we also need to take into account that $N_i(s_i - s_{i-1}) = N_i(ct_i - ct_{i-1}) = cM_i(t_i - t_{i-1})$. That is, using all this information, $cU(f(cx), P) = U(f(x), Q)$ and likewise $cL(f(cx), P) = L(f(x), Q)$. Now we can use the one-to-one correspondence outlined above between partitions of $[a, b]$ and $[ca, cb]$; from these equations we find that $\sup cL(f(cx), P) = c \sup L(f(cx), P) = \sup L(f(x), Q)$ and inf $cU(f(cx), P) = c \inf (f(cx), P) = \inf U(f(x), Q)$. But since $f$ is integrable over $[ca, cb]$ the third and sixth terms are equal, so the second and fifth are equal too. Thus by definition $\int_a^b f(x) \, dx$ exists, and

$$\int_{ca}^{cb} f(x) \, dx = c \int_{a}^{b} f(cx) \, dx$$

holds true.

13.20

Suppose that $f$ is nondecreasing on $[a, b]$. Notice that $f$ is automatically bounded on $[a, b]$, because $f(a) \leq f(x) \leq f(b)$ for $x \in [a, b]$.

(a) If $P = \{t_0, \ldots, t_n\}$ is a partition of $[a, \bar{b}]$, what is $L(f, P)$ and $U(f, P)$?

Since $f$ is nondecreasing, $f(t_{i-1}) \leq f(x)$ for all $x \in [t_{i-1}, t_i]$; likewise $f(t_i) \geq f(x)$ for all $x \in [t_{i-1}, t_i]$. So we have the easy formula

$$L(f, P) = \sum_{i=1}^{n} f(t_{i-1})(t_i - t_{i-1}) \text{ and } U(f, P) = \sum_{i=1}^{n} f(t_i)(t_i - t_{i-1}).$$

(b) Suppose that $t_i - t_{i-1} = \delta$ for each $i$. Prove that $U(f, P) - L(f, P) = \delta[f(b) - f(a)]$.

Using the above sums, we can simplify $U(f, P) - L(f, P)$ to the expression

$$\sum_{i=1}^{n} \delta f(t_i) - \sum_{i=1}^{n} \delta f(t_{i-1}),$$

which reduces to $\delta[f(t_n) - f(t_0)] = \delta[f(b) - f(a)]$.

(c) Prove that $f$ is integrable.

By Theorem 2, given $\varepsilon > 0$ we only need to show there is a partition $P$ of $[a, \bar{b}]$ so that $U(f, P) - L(f, P) < \varepsilon$. But simply take the partition described in part (b) with $\delta < \frac{\varepsilon}{f(b) - f(a)}$; then $U(f, P) - L(f, P) = \delta[f(b) - f(a)] < \varepsilon$.

(d) Give an example of a nondecreasing function on $[0, 1]$ which is discontinuous at infinitely many points.

The following function is discontinuous at every point $x = \frac{1}{n}$ for $n \in \mathbb{N}$, which is infinitely many. It is also clearly nondecreasing.

$$f(x) = \begin{cases} 
0, & x = 0 \\
\frac{1}{n}, & x \in \left(\frac{1}{n+1}, \frac{1}{n}\right], \ n \in \mathbb{N}
\end{cases}$$