23.3

(a) Show that \( \int_0^\infty e^y/y^k \, dy \) exists, by considering the series \( \sum_{n=1}^\infty (e/n)^n \).

For \( \int_1^\infty e^y/y^k \, dy \), we can apply the integral test. Indeed, \( f(y) = e^y/y^k > 0 \)
clearly, and by taking derivatives we see it is decreasing. That is, \( f'(y) = [e^y/y^k]' = \frac{e^y \log(e/y)}{y^{k+1}} = \frac{\log(e/y)}{y^k} + \frac{e^y}{y^{k+1}} \)\( \log(e/y) \) is decreasing.

Thus the integral exists if and only if \( \sum_{n=1}^\infty (e/n)^n \) converges. But we consider the ratio test (as the summands are positive);

\[
\lim_{n \to \infty} \frac{(\frac{n+1}{n})^{n+1}}{\frac{n}{(n+1)^{n+1}}} = \lim_{n \to \infty} \frac{e^{n+1}}{e^n} = \frac{e}{n+1}.
\]

But \( \lim_{n \to \infty} \frac{n}{e^n} = 0 \) by a homework you did in Chapter 18. So we see that

(b) Show that

\[
\sum_{n=2}^\infty \frac{1}{\log(n)^{\log(n)}}
\]

converges, by using the integral test.

Now clearly \( \frac{1}{\log(n)^{\log(n)}} \) is positive for \( n > 1 \); also, since log grows, however slowly, so does \( \log(n)^{\log(n)} \). Then by the integral test this series converges only if \( \int_2^\infty \frac{\ln(x)}{\log(x)^{\log(x)}} \) exists. We use the substitution \( y = \log(x) \), with \( dx = e^y \, dy \), or \( dx = e^y \, dy \). Then our integral is \( \int_0^\infty e^y \, dy \), which by part (a) exists.

(c) Show that

\[
\sum_{n=2}^\infty \frac{1}{\log(n)^{\log(n)}}
\]

diverges, by using the integral test.

Again, this is certainly positive for \( n > 1 \), and in addition for \( n \geq e \) we also have all exponents greater than or equal to one, so the denominator is growing (so the summands are shrinking). One can also use derivatives to see this. In any case, we apply the integral test for \( f(x) = \frac{1}{\log(x)^{\log(x)}} \); we use the substitution \( y = \log(x) \),

with \( dx = e^y \, dy \). This yields \( \int_2^\infty \frac{\ln(x)}{\log(x)^{\log(x)}} = \int_0^\infty \frac{e^y}{y^{\log y}} \, dy \). Note that \( y^{\log y} = e^{(\log y) \log y} = e^{(\log y)^2} \). So we consider the integral \( \int_0^\infty \frac{e^{dy}}{y^{\log y}} = \int_0^\infty e^{y^{log y}} \, dy \).

By \( \log^2 y \) is increasing \( (y - \log^2 y)' = 1 - 2 \frac{\log y}{y^2} \) which is greater than zero

But \( y - \log^2 y \) is increasing \( (y - \log^2 y)' = 1 - 2 \frac{\log y}{y^2} \) which is greater than zero

for \( y > e \), so \( e^{y - \log^2 y} \) certainly does not go to zero as is needed for the integral to
converge. By the integral test, the series converges only if the integral exists, which it does not.

24.2

For the following sequences of functions, determine the pointwise limit (if it exists) and whether the convergence is uniform.

(ii) \( f_n(x) = \frac{nx}{n+x} \) on \([0, \infty)\).

By Th. 1 of Ch. 22, the pointwise limit is the same as that as \( y \to \infty \) of \( \frac{xy}{x+y} \), which by L'Hopital's rule is \( \lim_{x \to \infty} x = x \). For \( \epsilon > 0 \), suppose we have an \( N \) as in the definition of uniform convergence. Then \( |f_n(x) - f(x)| < \epsilon \) for \( n \geq N \). But this absolute value is \( \frac{|nx-(1+n)x|}{1+n+x} = \frac{x^n}{1+n+x} \); however, the limit of this expression as \( x \to \infty \) is infinite, as we see by considering \( \frac{1+x}{1+n+x} \). Hence we can make this greater than \( \epsilon \), a contradiction. So it does not converge uniformly.

(iv) \( f_n(x) = \sqrt{x^2 + 1/n^2} \) on \( \mathbb{R} \).

Since \( 1/n^2 \) goes to zero, it is clear that \( f_n(x) \) approaches \( \sqrt{x^2} \). So we consider first for some small \( a > 0 \), \( -a, a \). Then as here \( |x| \leq a \), \( \sqrt{x^2 + 1/n^2} - \sqrt{x^2} \leq \sqrt{x^2 + 1/n^2} + \sqrt{x^2} \leq \sqrt{a^2 + 1/n^2} + \sqrt{a^2} \), so if we find an \( N \) such that for \( n \geq N \), \( \sqrt{a^2 + 1/n^2} + \sqrt{a^2} < \epsilon \), then for all \( x \) in this interval it is true. Note that we had better choose \( a \) small enough so that this inequality is true for some \( n! \). Then we do some algebraic manipulations: \( \sqrt{a^2 + 1/n^2} + \sqrt{a^2} < \epsilon \) if and only if \( \sqrt{a^2 + 1/n^2} < \epsilon - \sqrt{a^2} \); since quantities are positive, this is true if and only if \( a^2 + 1/n^2 < \epsilon^2 + a^2 - 2a|a| \). That is, again noting that we have chosen \( a \) such that the quantities are positive, if and only if \( 1/n^2 < \epsilon^2 - 2a|a| \), then \( n > \frac{1}{\sqrt{\epsilon^2 - 2|a|}} \). So we take the first such integer \( n \) and set it equal to \( N \). So we have uniform convergence on \([-a, a]\). Note that as we have each of the above steps if and only if, there is no need to plug it back in - we know it works.

But for \((-, \infty, a] \cup [a, \infty)\), we need a different method. Since \( |x| \geq a \), we have \( \sqrt{a^2 + 1} + \sqrt{a^2 n^2} \leq \sqrt{x^2 + 1} + \sqrt{x^2 n^2} \), whence \( 1/(\sqrt{x^2 + 1} + \sqrt{x^2 n^2}) \leq 1/(\sqrt{a^2 + 1} + \sqrt{a^2 n^2}) \). Now we write

\[
\frac{\sqrt{x^2 + 1} - \sqrt{x^2}}{x^2 + 1/n^2 - x^2} = \frac{\sqrt{x^2 + 1} - \sqrt{x^2}}{n} = \frac{\sqrt{x^2 + 1} - \sqrt{x^2}}{n} \cdot \frac{\sqrt{x^2 + 1} + \sqrt{x^2}}{\sqrt{x^2 + 1} + \sqrt{x^2}} = \frac{1}{\frac{n}{\sqrt{x^2 + 1} + \sqrt{x^2} n^2}}.
\]

By a comment above, this is less than or equal to \( \frac{1}{n(\sqrt{a^2 + 1} + \sqrt{a^2 n^2})} \). Thus if we can make this quantity less than \( \epsilon \) for all \( n \) greater than some \( N \), the sequence will stay within \( \epsilon \) of the function we are converging to, so we will have uniform convergence. But again we do the algebraic manipulations as above and find that we only need \( N > \frac{1}{\sqrt{a^2 + 2|a|}} \). Finally, for the whole real line we take the greater of the two values of \( N \); certainly any \( n \geq N \) in that case will be large enough for both the small interval around 0 and the majority of the real line analyzed here. Where!

(vi) \( f_n(x) = \sqrt{x + 1/n - x} \) on \( \mathbb{R} \).

First note that this is meaningless for \( x < 0 \), so we only consider \([0, \infty)\). Since \( 1/n \to 0 \) as \( n \to \infty \), this converges to \( f(x) = 0 \). Again, first consider some small \( a > 0 \) and the interval \([0, a]\). Then again \( |x| \leq a \); \( |\sqrt{x + 1/n} - \sqrt{x}| \leq \sqrt{x + 1/n + \sqrt{x}} \leq \sqrt{a + 1/n + \sqrt{a}} \), so if we find an \( N \) such that for \( n \geq N \),
\[ \sqrt{a + 1/n} + \sqrt{a} < \epsilon, \] then for all \( x \) in this interval it is true. Note that we had better choose \( a \) small enough so that this inequality is true for some \( n \! \). Then we do some algebraic manipulations: \( \sqrt{a + 1/n} + \sqrt{a} < \epsilon \) if and only if \( \sqrt{a + 1/n} < \epsilon - \sqrt{a} \); since quantities are positive, this is true if and only if \( a + 1/n < \epsilon^2 - a - 2\epsilon \sqrt{a} \). That is, again noting that we had chosen \( a \) such that the quantities are positive, if and only if \( 1/n < \epsilon^2 - 2\epsilon \sqrt{a}, \) or \( n > \frac{1}{\epsilon^2 - 2\epsilon \sqrt{a}}. \) So we take the first such integer \( n \) and set it equal to \( N. \) So we have uniform convergence on \([0,a].\) Note that as we have each of the above steps if and only if, there is no need to plug it back in - we know it works. But for \([a,\infty),\) we need a different method. Since \( x \geq a, \) we have \( \sqrt{a + 1/n} + \sqrt{a} \leq \sqrt{x} + 1/n + x, \) when \( 1/(\sqrt{x} + 1/n + x) \leq 1/(\sqrt{a + 1/n + a}). \) Now we write

\[
\sqrt{x + 1/n} - \sqrt{x} = \frac{\sqrt{x_n} + 1 - \sqrt{x_n}}{\sqrt{n}} = 1
\]

By a comment above, this is less than or equal to \( \frac{1}{\sqrt{n} (\sqrt{x + 1/n} + \sqrt{x})}. \) Thus if we can make this quantity less than \( \epsilon \) for all \( n \) greater than some \( N, \) the sequence will stay within \( \epsilon \) of the function we are converging to, so we will have uniform convergence. But again we do the algebraic manipulations as above and find that we only need \( N > \frac{1}{\epsilon^2 - 2\epsilon \sqrt{a}}. \) Finally, for the whole real line we take the greater of the two values of \( N;\) certainly any \( n \geq N \) in that case will be large enough for both the small interval around \( 0 \) and the majority of the real line analyzed here.

(viii) \( f_n(x) = n(\sqrt{x + 1/n} - \sqrt{x}) \) on \([0,\infty)\)

First we need to see what this converges to pointwise. Again using Theorem 22.1, we want to find \( \lim_{y \to \infty} y(\sqrt{x + 1/y} - \sqrt{x}); \) but as the roots go to zero and so does \( 1/y, \) we may use L'Hopital's rule to see that we get \( \lim_{y \to \infty} \frac{1/(2x \sqrt{x + 1/n})}{-1/n} = \frac{1}{2\sqrt{x}}. \) This is of course for nonzero \( x; \) for \( x = 0, \) we simply have \( f_n(0) = n\sqrt{1/n}, \) which does not even converge. Thus we clearly cannot have uniform convergence!

2.4

(iii) Find the infinite sum \( \sum_{n=0}^{\infty} \frac{x^n}{2^n} - \frac{3^n}{5^3} + \frac{x^3}{4^3} - \frac{x^3}{5^3} + \cdots \) for \( |x| < 1. \)

The term by term derivative of this is \( x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \) which converges to \( \log(1 + x) \) for \( |x| < 1. \) Since on any closed interval in \((-1,1)\) we have uniform convergence by Theorem 6, then on such an interval by the Corollary of this chapter since all the monomial terms of the second series and log are integrable on this interval, the integral (over that closed interval) is the series of the integrals of the monomials, which is of course precisely the infinite sum in question. But that means that it sums to the antiderivative on that closed interval. Since any number in \((-1,1)\) can be found in such a closed interval, we thus see that \( \frac{x^2}{2} - \frac{x^3}{3^3} + \frac{x^3}{4^3} - \frac{x^5}{5^3} + \cdots \) for \( |x| < 1 \) is \( (1 + x) \log(1 + x) - x. \)

2.5

(iv) Evaluate \( \sum_{n=0}^{\infty} \frac{2^n}{n!}. \)

I know of two ways to evaluate this. One can remove 1/2 from each summand, and then notice that one has the series with terms \( nx^{n-1} \) for \( x = 1/2, \) which is the derivative of \( x^n; \) then using some theorems from the chapter and the fact that a geometric series with ratio less than 1 converges, this can be evaluated (this is just a
sketch of that proof, of course). Another interesting way to do this is the following. The first term of the series is 0, of course. Then we note that since all the terms are positive, if it converges, it converges absolutely, so rearrangements are possible. I write $\frac{x}{n}$ as $\sum_{j=1}^{n} \frac{x}{n}$. Then we have $\sum_{n=1}^{\infty} \sum_{j=1}^{n} \frac{x}{n}$. Now since we have precisely $n$ of each $1/2^n$, we can do the following reordering (check to see there are the same number of copies of each term): $\sum_{n=1}^{\infty} \frac{1}{2^n-1} \left( \sum_{m=1}^{\infty} \frac{1}{2^n} \right)$; as each of the inner sums is well-known to be $\frac{1}{1-1/2} = 1$, we have $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{1-1/2} = 2$ as the sum.

24.8

Prove that the series

$$\sum_{n=1}^{\infty} \frac{x}{n(1 + nx^2)}$$

converges uniformly on $\mathbb{R}$.

We use the M-test. First we need to find any old bound at all on $|a_n|$; one usually does this by taking derivatives. We note that the function is odd ($a_n(-x) = -a_n(x)$). We see that the derivative, using the quotient rule, is $\frac{n(1 + nx^2)^2 - 2nx^2}{(1 + nx^2)^2}$, which simplifies to $\frac{1-nx^2}{n(1+nx^2)^2}$. This has zeros at $x = \pm \sqrt{1/n}$; further, as the derivative is positive for $x \in (-\sqrt{1/n}, \sqrt{1/n})$ and negative for $|x| > \sqrt{1/n}$, the maximum for $|a_n|$ occurs at these critical points (recall it is odd). Hence $|a_n| \leq \frac{\sqrt{1/n}}{n(1+nx^2)} = \frac{1}{2nx^2}$. But now for any given $n$, we let this last expression be $M_n$; then by the criterion given in Chapter 23 by the integral test, the series $\sum_{n=1}^{\infty} M_n$ is summable; hence by the Weierstrass M-test the series in question converges uniformly on the whole real line.

24.9

Prove that the series

$$\sum_{n=0}^{\infty} 2^n \sin \frac{1}{3^n x}$$

converges uniformly on $[a, \infty)$ for $a > 0$. By considering the sum from $N$ to $\infty$ for $x = 2/(3^N)$, show that the series does not converge uniformly on $(0, \infty)$.

Again we will use the M-test. The derivative of this $a_n$ is $-\frac{2^n}{3^n x} \cos \frac{1}{3^n x}$. Now this is unbounded for $x \to 0$, but we are only dealing with some $a > 0$. No matter how small $a$ is, there is an $N$ such that $3^n a > 2/\pi$ for all $n \geq N$ (and of course for all $x > a$), so that for these $n$ the derivative above is always negative. Then the maximum of $a_n$, for these $n$, occurs at the endpoint, namely at $a$, where the value is $2^n \sin \frac{1}{3^n a}$. But since we have established that $3^n a > 2/\pi$, we recall that (by checking derivatives) $\sin h < h$ in this interval, and furthermore $\sin$ is positive here, so $a_n < \left( \frac{2}{3^n a} \right)^2$. We let the latter be $M_n$, and by convergence of geometric series of ratio less than 1 we have that the series $\sum_{n=1}^{\infty} M_n$ converges; thus $\sum_{n=N}^{\infty} 2^n \sin \frac{1}{3^n x}$ converges uniformly to the function $\sum_{n=N}^{\infty} 2^n \sin \frac{1}{3^n x}$. Then if we add $\sum_{n=0}^{N-1} 2^n \sin \frac{1}{3^n x}$ to both series this does not affect uniform convergence, since we added identical amounts to both series.

To see that we do not have uniform convergence on $(0, \infty)$, suppose that for $\epsilon = 1$ we have found such an $N$. Then consider the sum $\sum_{n=N}^{\infty} 2^n \sin \frac{1}{3^n (2/\epsilon - \delta)}$, as in the hint. Each term is $2^n \sin \left( \frac{\pi}{3^n (2/\epsilon - \delta)} \right)$, which has the value $2^n$. This series certainly
isn’t staying within \( \epsilon = 1 \) of anything except infinity. So we were mistaken in our choice of \( N \), contradiction, and we do not have uniform convergence.

**24.12**

(a) Suppose that \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) converges for all \( x \) in some interval \((-R, R)\), and that \( f(x) = 0 \) for all \( x \in (-R, R) \). Prove that each \( a_n = 0 \).

But we know by the formula on page 507 that a convergent power series at zero is the Taylor series at zero of the function it defines, so \( a_n = \frac{f^{(n)}(0)}{n!} \). But for any point \( x \) in this interval, there is some \( \epsilon \) such that \( x - \epsilon \) and \( x + \epsilon \) are in the interval, and so all derivatives are zero (as \( f \) is constant) on \([x - \epsilon, x + \epsilon]\). Then all the \( a_n = 0 \), since this is true for \( x = 0 \).

(b) Suppose we know only that \( f(x_n) = 0 \) for some sequence \( \{x_n\} \) with \( \lim_{n \to \infty} x_n \) being 0. Prove again that each \( a_n = 0 \).

Now since the series converges, it converges uniformly on any closed interval in the open interval (by Theorem 6). But since each partial sum is a polynomial and so continuous, by the big Corollary the function converged to is also continuous, so by Theorem 22.1 \( \lim_{n \to \infty} f(x_n) = 0 \) and \( f(0) = a_0 = 0 \). Suppose we have this property for some sequence converging to zero for \( f^{(k)}(0) = a_k \); by repeated application of Rolle’s Theorem to the intervals \([x_n, x_{n+1}]\), we now come up with a new sequence \( \{y_n\} \), each \( y_n \in [x_n, x_{n+1}] \), which also converges to zero, however at which \( f^{(k+1)}(y_n) = 0 \). By Theorem 6, the termwise derivative of the power series converges uniformly to derivative of the function converged to, and Spivak has already noted all derivatives exist for us on \((-R, R)\). Then we may use the big Corollary again to see that \( \lim_{n \to \infty} f^{(k+1)}(y_n) = 0 \) and \( f^{(k+1)}(0) = a_{k+1} = 0 \). Then by induction, we are done.

(c) Suppose that \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) and \( g(x) = \sum_{n=0}^{\infty} b_n x^n \) converge for all \( x \) in some interval containing zero and that \( f(t_n) = g(t_n) \) for some sequence \( \{t_n\} \) converging to zero. Show that \( a_n = b_n \) for each \( n \).

We merely consider \((f-g)(x)\), which will perform (by page 505) be the termwise convergent sum of the individual power series. Then we use part (b) to show that each coefficient \( a_n - b_n = 0 \), so the result follows. I would probably expect you to actually write the sums in question again, but so that I can get this to you in a timely fashion I will leave that tacit.

**24.28**

(a) Suppose that \( \{f_n\} \) is a sequence of continuous functions on \([a, b]\) that converges uniformly to \( f \). Prove that if \( x_n \) approaches \( x \), then \( f_n(x_n) \) approaches \( f(x) \).

Take some \( \epsilon > 0 \). Since \( x_n \) approaches \( x \), for some \( N, |x_n - x| < \frac{\epsilon}{2} \) for \( n > N \). Similarly, by uniform convergence, for some \( M \) \( |f_n(y) - f(y)| < \frac{\epsilon}{2} \) for all \( y \in [a, b] \) and \( n > M \). So for \( n > \max(M, N) \), \( |f_n(x_n) - f(x)| \leq |f_n(x_n) - f_n(x)| + |f_n(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \). So it is proved.

(b) Is the statement true without assuming that the \( f_n \) are continuous?

No. Take the interval \([0, 2]\) and the functions \( f_n(x) = 1 \) for \( x < 1 \) and \( f_n(x) = 0 \) for \( x > 1 \). Obviously this converges uniformly (\( N=1 \), in fact). But if we let \( x_n = \frac{n+1}{n} \), which converges to \( x = 1 \), then \( f_n(x_n) \) converges to \( 1 \), not to \( f(1) = 0 \).

(c) Prove the converse of part (a): If \( f \) is continuous on \([a, b]\) and \( \{f_n\} \) is a sequence with the property that \( f_n(x_n) \) approaches \( f(x) \) whenever \( x_n \) approaches \( x \), then \( f_n \) converges uniformly to \( f \) on \([a, b]\).

Assume not. Then for some \( \epsilon > 0 \), for each \( n \) there is a point \( x_n \) such that \( |f_n(x_n) - f(x_n)| > \epsilon \) for (otherwise for some large enough \( n \) all of them would be
less than \( \epsilon \), which would imply uniform convergence). This is of course a bounded sequence as we are on a closed interval; hence by Bolzano-Weierstrass there is a subsequence \( a_k \) which converges to some \( x \). But then by the property in question, \( f_k(a_k) \) approaches \( f(x) \), i.e. for some \( N, k > N \) implies \( |f_k(a_k) - f(x)| < \frac{\epsilon}{2} \). However, by continuity there is an \( M \) such that for \( k > M, |f(x) - f(a_k)| < \frac{\epsilon}{2} \) as well; then \( |f_n(a_k) - f(a_k)| \leq |f_k(a_k) - f(x)| + |f(x) - f(a_k)| < \epsilon \), a contradiction since each \( a_k = x_n \) for some \( n \). So we do have uniform convergence.

**Extra**

Conclude from the proof that \( f(1/N) \geq \frac{1}{8} \) (for \( f(x) = \sum_{n=0}^{\infty} \frac{nx^n}{1 + nx^n} \)) that \( f \) does not converge uniformly on \([0, \infty)\).

Now we can easily see that \( f(0) = 0 \). But for \( \epsilon < \frac{1}{8} \), this means that \( |f(x) - f(0)| > \epsilon \) for (infinitely many) \( |x| < \delta \) for any arbitrarily small \( \delta > 0 \). So \( f \) is not continuous. Thus, as all the partial sums are continuous, the convergence cannot be uniform (otherwise \( f \) would be too).