

Research Statement for Karl-Dieter Crisman

My current research is primarily in the mathematics of voting. This discipline lies at the interface of mathematics, political science, and economics, though it has connections to any field which could involve aggregation of preferences, such as statistics or psychology. It also is a proven, fertile field for undergraduate research.

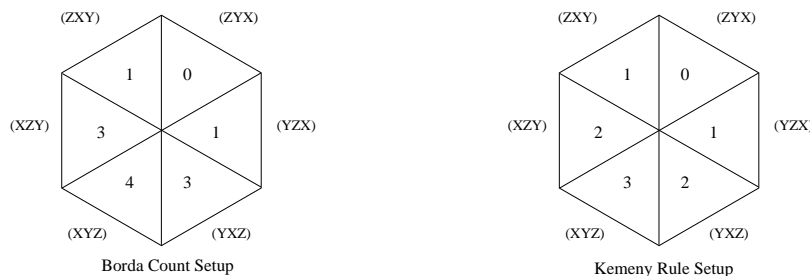
The math of voting, in its classical form, asks questions about what is and isn't possible with various voting methods; these include the usual plurality vote, a runoff system such as in television 'reality shows' or primaries, or weighted voting systems such as the Borda Count and those used in collegiate sports polls.

The seminal theorems are intriguing even to the non-expert. Arrow proved in the early 50's (essentially) that, for three or more candidates, there does not exist a voting system which fulfills the most basic criteria of fairness (symmetry, really) and simultaneously agrees with all pairwise comparisons between candidates. Twenty years later, Gibbard and Satterthwaite proved that there also does not exist such a system which cannot be manipulated by insincere voting (for an example from the most recent presidential campaign, such as an Obama supporter supporting Clinton in the primary as more 'electable').

Before describing current work, we need a little notation. The basic concept is that of a *voting profile*, which is a collection of voters along with their preferences. For instance, the set $\{4, 3, 1, 0, 0, 0\}$ is often thought of as representing a profile where four voters prefer $A \succ B \succ C$, three support $A \succ C \succ B$, and one has the preference order $C \succ A \succ B$. If we think of a profile as a vector, we can think of the set of *all* profiles as a vector space.

It turns out that there are geometric decompositions of this profile space which are highly profitable in analyzing different methods and their relative strengths (most notably done by D. Saari), and it further turns out that one can re-encode these ideas algebraically in terms of representation theory (e.g., as done by M. Orrison and his students); the portion of a profile space which is subject to the fewest paradoxes is called the *Basic* or *Borda* portion. My work tries to answer new questions from these algebraic/geometric points of view, including the following:

- Certain non-parametric statistical tests for uniformity may be interpreted as first turning data from independent samples into a profile, next applying some voting method to the profile, and finally getting a test statistic from that information. Saari, Haunsperger, and Bargagliotti have all contributed to trying to understand the relation between the statistics and the voting. My goal is to completely understand the structure of the set (importantly, not a vector space) of profiles which arises from this, in particular the Basic portion. A student of mine and I have submitted a paper about this, and one of my lemmas from that paper should serve to help unravel the combinatorics of this question in the future.
- Many voting procedures involve giving points to all candidates in a voter's preference order, weighted by how much they are preferred (such as the in the Borda Count). However, in a *social welfare function* the goal is not to choose a winner, but rather to choose a total ranking, so many theorems about maximally symmetric methods no longer apply. Instead, one can choose to give points to *the orders*, not just the candidates (this idea is due to Zwicker and Conitzer, in different forms). The natural geometric object is no longer the simplex, but the *permutahedron*, which in the case of three candidates is just the regular hexagon.



It turns out that one can decompose profiles in such a way to clarify the role of the Basic component, and I have done so (preprint in preparation). The representation theory of the symmetry group of the permutahedron actually gives a direct computation of exactly how many such social welfare functions there are satisfying various fairness criteria; in particular, there is a new connection between the Borda Count and another important method called the Kemeny Rule, as members of a one-dimensional family of voting methods. Future work would extend these methods to other voting-theoretic contexts (such as cyclic preferences) and seek explicit connections with analogous geometric results.

- Some states are considering (or have passed) laws to assign all their *state* electors to the *national* plurality winner, in an effort to avoid paradoxes such as the Hayes/Tilden or Bush/Gore elections, where a plurality of US voters selected the loser in the electoral college. What paradoxes or disenfranchisements are likely to arise from such a move, in a theoretical sense?

I also have done some work in other fields. Questions of interest include:

- Music theory is replete with areas friendly to mathematical analysis. Recently, much good work has been done in geometric and algebraic representations of music, notably in Hook's thesis on group actions on chord spaces, and the Callender/Quinn/Tymoczko space (in fact, an orbifold) of unordered pitch-class sets. However, there are many remaining natural questions mathematicians (and their undergraduate students) can attack. My current interest is in reinterpreting and extending some of the above ideas (especially Quinn's thesis) more explicitly in terms of group representations, where they should most naturally fit.
- My graduate work in algebraic geometry was in generalizations of Bloch's higher Chow groups $CH^p(k, n)$. One can consider these as being groups of cycles relative to the arrangement of hyperplanes given by the algebraic simplex; hence, for a general arrangement \mathcal{A} , with a group $A^n(k, \mathcal{A})$ of zero-cycles on the complement of \mathcal{A} , one can ask for a 'good' subgroup $B^n(k, \mathcal{A})$ such that

$$AH^n(k, \mathcal{A}) := A^n(k, \mathcal{A})/B^n(k, \mathcal{A})$$

could be called the Chow group of zero-cycles *relative to* \mathcal{A} . My thesis characterized these appropriately and extended to a certain normal crossing divisor case, where $AH^n(k, \mathcal{A}) = K_n^M(k)$ (Milnor K-theory). For the worst degeneracy possible, the expected answer is that we recover the DeRham-Witt complex of Madsen et al. (this is true in the simplest case, where one gets algebraic differentials $\Omega_{k/\mathbb{Z}}$), and I have a provisional Witt vector action on the appropriate group in this case.