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Homogeneous Coordinates

They work, but where do they come from?

Jonathan Senning

jonathan.senning@gordon.edu

Outline

- Two-dimensional computer graphics
- Translation – necessary but nonlinear
- Homogeneous coordinates
- Affine geometry and spaces
- Frames
- Homogeneous coordinates again
- Perspective projections

Two-dimensional computer graphics

- Provides an interesting motivational example
- Illustrates linear transformations
- Early exposure to idea of isomorphisms
- Three key operations:
  - scaling
  - rotation
  - translation

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Scaling

To scale the *x* coordinate by *α* and the *y* coordinate by *β* we can use the scaling matrix defined by

$$S(\alpha, \beta) = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

Applying the transformation scales the vector

$$\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha x \\ \beta y \end{bmatrix}$$

Rotation

The matrix that implements a counterclockwise rotation about the origin by the angle θ is

$$R(\theta)=\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}$$

Applying this gives

$$\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}x\cos(\theta)-y\sin(\theta)\\ x\sin(\theta)+y\cos(\theta)\end{bmatrix}$$

Shearing

Shearing can also be easily implemented with a matrix

$$H(\gamma,\delta)=\begin{bmatrix}1 & \gamma\\ \delta & 1\end{bmatrix}$$

Notice that x is increased by a factor of y; shearing, like rotation, is relative to the origin

$$\begin{bmatrix}1 & \gamma\\ \delta & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}x+\gamma y\\ y+\delta x\end{bmatrix}$$

Translation

Unfortunately translation cannot be implemented with matrix-vector multiplication.

Here comes the magic...

Homogeneous Coordinates

This magic is called "homogeneous coordinates" and (from a student's perspective) consists of appending a 1 to the *x* and *y* coordinates.

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Homogeneous Coordinates

The magic of homogeneous coordinates is twofold:

1. The matrices of existing linear transformations can easily be extended to work with homogeneous coordinates

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \rightarrow \begin{bmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Homogeneous Coordinates

2. Translation can be implemented as matrix-vector multiplication

$$\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+h \\ y+k \\ 1 \end{bmatrix}$$

Little justification is offered for where homogeneous coordinates come from; the justification is usually "they just work."

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Affine Spaces

Definition: An affine space consists of a vector space *V* and a set of points *P* such that

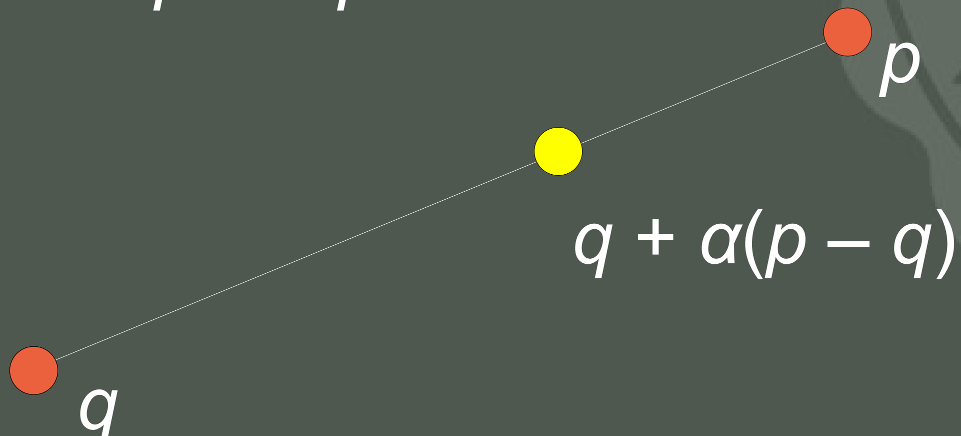
1. the difference of any two points from *P* is a vector in *V*,
2. given any point *p* from *P* and vector **v** from *V*, the sum *p* + **v** is a point in *P*.

Affine Spaces

Suppose p and q are points from P. Then

1. $\alpha(p - q)$ is a vector in V, and

2. $q + \alpha(p - q)$ is a point in P that lies along the line between q and p.



Affine Combinations

This last expression, $q + \alpha(p - q)$, yields a point according to the axioms for affine spaces. If we formally rearrange this we can obtain

$$\alpha p + (1 - \alpha)q$$

or

$$\alpha p + \beta q$$

where $\alpha + \beta = 1$.

This is defined to be an *affine combination*; an affine combination of points yields a point.

Frames

A *frame* for an affine space $A=(V,P)$ is analogous to a basis for a vector space.

A frame consists of two things:

- a basis B for the vector space V and
- a point o from P.

Every vector in V is a linear combination of vectors in B.

Every point in P can be obtained by adding a vector from V to o.

Frames

- A frame for an affine space with an n-dimensional vector space contains $n+1$ elements.
- A frame allows us to locate and orient an n-dimensional vector space relative to another n-dimensional vector space.

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Coordinate Axiom

To make use of a frame for an affine space we need one additional axiom. The *coordinate axiom* is quite simple and states:

For every point p in an affine space A

- $0p$ is defined to be the zero vector $\mathbf{0}$ in A, and
- $1p$ is defined to be the point p.

Frames and the Coordinate Axiom

Suppose the frame for an affine space *A* is given by a basis for *V*

$$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$$

and a point *o* from *P*. Because of the coordinate axiom, any vector **v** in *A* can be written as

$$\mathbf{v} = \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_n \mathbf{u}_n + 0\,o$$

for suitable choice of constants α_i.

Frames and the Coordinate Axiom

Similarly, any point *p* in *A* can be written as

$$p = \beta_1 \mathbf{u}_1 + \beta_2 \mathbf{u}_2 + \cdots + \beta_n \mathbf{u}_n + 1\, o$$

for suitable β_i.

Once a frame is defined, we can describe coordinates of elements in an affine space relative to the frame.

As in the case of bases for vector spaces, we assume that elements of the frame are ordered and we require that the point *o* appears last.

Frame Coordinate Vectors

The coordinate vectors of the vector **v** and the point *p are*

$$[\mathbf{v}]=\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\\ 0\end{bmatrix} \qquad [p]=\begin{bmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_n\\ 1\end{bmatrix}$$

Notice that the coordinate vector of a vector ends with 0 while the coordinate vector of a point ends with 1.

Homogeneous Coordinates Again

The frame coordinate vectors are exactly the same as the homogeneous coordinates we've already seen!

An interesting project at this point would be to have students derive the transformation matrices for scaling, rotation, and translation by finding suitable frames and the corresponding change-of-frames matrices.

Another View of Homogeneous Coordinates

The set of vectors given by $\begin{bmatrix} x & y & 1 \end{bmatrix}^T$ can be viewed as points on a plane parallel to the *xy*-plane but translated away from the origin by 1 unit along the *z* axis.

We can extend the definition of homogeneous coordinates so that $\begin{bmatrix} x & y & 1 \end{bmatrix}^T$ is equivalent to all vectors $\begin{bmatrix} x' & y' & z' \end{bmatrix}^T$ when $x = x'/z'$ and $y = y'/z'$.

Another View of Homogeneous Coordinates

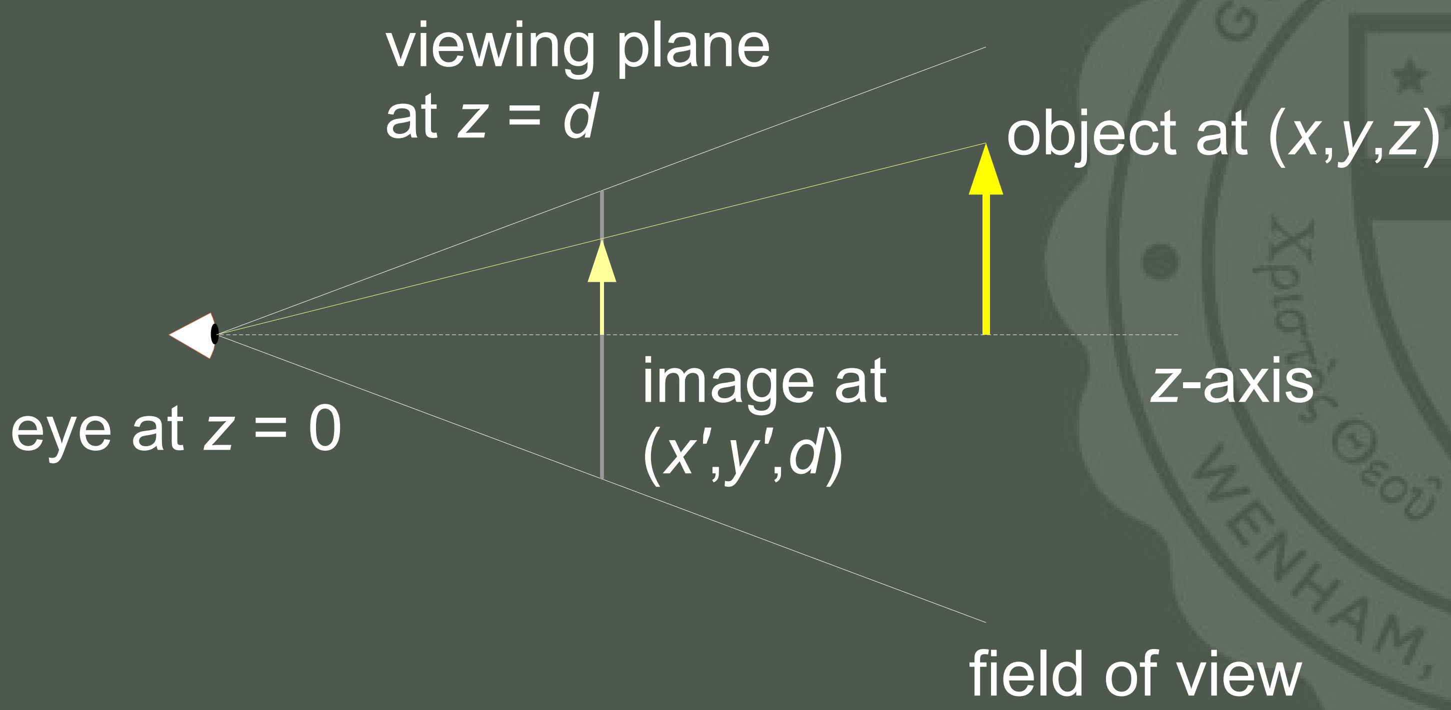
This means that homogeneous coordinates define an surjection of R^{n+1} onto an *n*-dimensional subspace of R^{n+1}.

Most computer graphics hardware implements the nonlinear scaling operation that normalizes the last coordinate as part of the pipeline that all points pass through. This is called *perspective division*.

From now on we'll be using 3-dimensions...

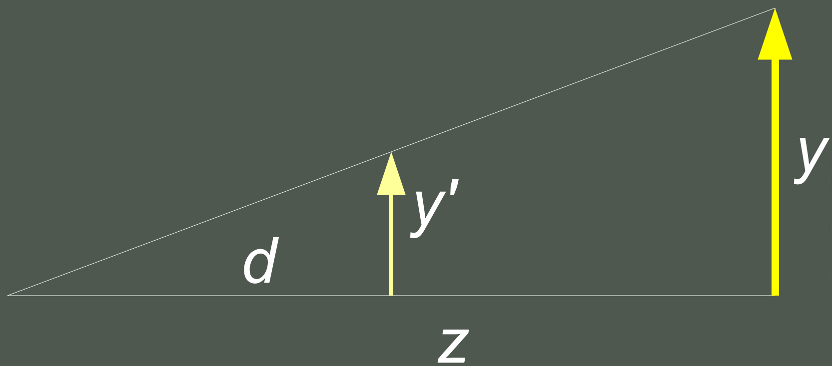
Perspective Projection

Cross section of perspective projection: *xy*-plane



Perspective Projection

From similar triangles we know $y'/y = z/d$ which gives $y' = y/(z/d)$.



Repeating these steps we can also find $x' = x/(z/d)$.

Perspective Projection

We need to perform the mapping

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} \frac{x}{z/d} \\ \frac{y}{z/d} \\ d \\ 1 \end{bmatrix}$$

This nonlinear operation can (almost) be done with a single matrix-vector multiplication.

Perspective Projection

Consider the following operation

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ z/d \end{bmatrix}$$

If *perspective division* is automatically performed by the graphics system, we obtain the desired vector for the image point.

Summary

- Frames provide a useful tool for modern computer graphics.
- Students easily pick up these concepts once they are comfortable with vector spaces and bases.
- Homogeneous coordinates arise naturally when affine spaces and frames are used, combating the “they just work” rationale for their use.
- Homogeneous coordinates make other important operations easy to implement in a modern graphics system.