Double Integrals

Problem: Find the volume of the region above $R$ and below $z = f(x, y)$.

Basic idea: Split the volume into many small rectangular prisms and sum the volume of these prisms.

We will partition $R$, a region of the $xy$-plane, into rectangles. These small rectangles need not be square: $\Delta A_i = \Delta x_i \Delta y_i$.

If the partition is called $P$, we can define the norm of the partition $\|P\|$ to be the largest diagonal length of any rectangle in the partition. Then as $\|P\| \to 0$, the lengths of the sides of each rectangle must go to zero, and all $\Delta A_i \to 0$ as well.

To find the volume of a typical prism, we take $(x_i, y_i)$ to be a point in the $i$th rectangle of $P$. Then $f(x_i, y_i)$ is the height of the prism.

$$\Delta V_i = f(x_i, y_i) \Delta A_i$$
Double Integrals

If we sum over all such prisms we will approximate the desired volume

\[ V \approx \sum_{i=1}^{n} f(x_i, y_i) \Delta A_i \]

As \( n \to \infty \) we see \( n \to \infty \), \( \Delta x_i \to 0 \), \( \Delta y \to 0 \) and

\[ \sum_{i=1}^{n} \Delta A_i \to \text{Area of } R \]

Also \( \sum_{i=1}^{n} \Delta V_i \to V \). This leads to the definition.

Def: For any function \( f(x, y) \) defined on a banded region \( R \in \mathbb{R}^2 \), we define the double integral of \( f \) over \( R \) by

\[ \iint_{R} f(x, y) \, dA = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i, y_i) \Delta A_i \]

provided the limit exists. We say \( f \) is integrable over \( R \).

Properties

1. \( \iint_{R} c f(x, y) \, dA = c \iint_{R} f(x, y) \, dA \)

2. \( \iint_{R} [f(x, y) + g(x, y)] \, dA = \iint_{R} f(x, y) \, dA + \iint_{R} g(x, y) \, dA \)

3. \( \iint_{R} f(x, y) \, dA = \iint_{R_1} f(x, y) \, dA + \iint_{R_2} f(x, y) \, dA \) if \( R = R_1 \cup R_2 \) and \( \emptyset \neq R_1 \cap R_2 \)
Double Integrals

Example: \( f(x,y) = 2 - x - y \)  
\( R: 0 \leq x \leq 1, \ 0 \leq y \leq 1 \)

\[ \Delta V_i = f(x_i, y_i) \Delta x_i \Delta y_i \]

The volume of the slice is approximated by

\[ V_{\text{slice}} \approx \left[ \sum_j f(x_j, y_j) \Delta y_j \right] \Delta x_i \]

where \( \Delta y_j \) are the lengths of \( dy \) along the slice.

As \( \Delta y \to 0 \) we have

\[ V_{\text{slice}} = \left[ \int_0^1 (2-x-y) \, dy \right] \Delta x_i \]

where \( x \) is fixed.

To find the total volume, we "add" up all the slices

\[ V = \sum_i \left[ \int_0^1 (2-x-y) \, dy \right] \Delta x_i \]

which, as \( \Delta x_i \to 0 \), we find

\[ V = \int_0^1 \left[ \int_0^1 (2-x-y) \, dy \right] \, dx \]

\[ = \int_0^1 \left( 2y - xy - \frac{y^2}{2} \right) \Big|_0^1 \, dx \]

\[ = \int_0^1 \left( 2 - x - \frac{1}{2} - 0 + 0 \right) \, dx \]

\[ = 2x - \frac{x^2}{2} - \frac{x}{2} \Big|_0^1 = 2 - \frac{1}{2} - 0 + 0 = \frac{1}{2} \]
Double Integrals

We could, of course, slice in the other direction, which would give

\[ V = \int_0^1 \int_0^1 (2-x-y) \, dx \, dy \]

\[ = \int_0^1 \left[ 2x - \frac{x^2}{2} - yx \right]_0^1 \, dy \]

\[ = \int_0^1 2 - \frac{1}{2} - y \, dy \]

\[ = \left. 2y - \frac{y^2}{2} \right|_0^1 = 2 - \frac{1}{2} - \frac{1}{2} = 1 \]

Thm (Fubini)

Suppose that \( f \) is integrable over the rectangle \( R = \{(x,y) \mid a \leq x \leq b, c \leq y \leq d\} \). Then

\[ \iint_R f(x,y) \, dA = \int_a^b \int_c^d f(x,y) \, dy \, dx = \int_c^d \int_a^b f(x,y) \, dx \, dy \]

These are called "iterated integrals." Note how the limits of integration match the differentials.

What if \( R \) is not a rectangle?

Ex. Evaluate \( \iint_R (3-x-\frac{1}{2}y) \, dA \) if \( R \) is the region bounded by \( y=0, y=4-2x, x=0 \)

\[ \text{\# Draw } R \]
The idea is the same as before, but now the length of the slice changes.

\[ dv = \text{volume of slice} = \left( \int_{0}^{4-2x} (3-x-y) \, dy \right) \, dx \]

\[ V = \int dv = \int_{0}^{2} \int_{0}^{4-2x} (3-x-y) \, dy \, dx \]

\[ = \int_{0}^{2} \left[ 3y - xy - \frac{y^2}{2} \right]_{0}^{4-2x} \, dx \]

\[ = \int_{0}^{2} \left[ 3(4-2x) - x(4-2x) - \frac{(4-2x)^2}{2} - 0 \right] \, dx \]

\[ = \int_{0}^{2} \left[ 12 - 6x - 4x + 2x^2 - \frac{16 - 16x + 4x^2}{4} \right] \, dx \]

\[ = \int_{0}^{2} \left( x^2 - 6x + 8 \right) \, dx \]

\[ = \left[ \frac{x^3}{3} - 3x^2 + 8x \right]_{0}^{2} \]

\[ = \frac{8}{3} - 12 + 16 - 0 = \frac{8}{3} + 4 = \frac{20}{3} \]
Double Integrals

Ex. Integrate \( f(x,y) = x^2y + x \) over \( R \) if \( R \) is the region bounded by \( y = 2x \) and \( y = x^2 \).

If we first integrate along \( y \) we find the lower and upper limits are \( y = x^2 \) and \( y = 2x \).

To find the limits on \( x \), we want to integrate over the furthest \( x \) extent of \( R \), in this case from \( x = 0 \) to \( x = 2 \).

\[
\iint_R x^2y + x \, dA = \int_0^2 \left( \int_{x^2}^{2x} x^2y + x \, dy \right) dx
\]

\[
= \int_0^2 x^2 \left[ \frac{y^2}{2} + xy \right]_{x^2}^{2x} \, dx
\]

\[
= \int_0^2 x^2 \left( \frac{4x^4}{2} + 2x^3 - \frac{x^6}{2} - x^3 \right) \, dx
\]

\[
= \int_0^2 2x^4 + 2x^3 - \frac{x^6}{2} - x^3 \, dx
\]

\[
= \left[ \frac{2}{5} x^5 + \frac{2}{3} x^3 - \frac{x^7}{14} - \frac{x^4}{4} \right]_0^2
\]

\[
= \frac{64}{5} + \frac{16}{3} - \frac{128}{7} - 4 - [0]
\]

\[
= \frac{64}{5} + \frac{16}{3} - \frac{64}{7} - 4
\]

\[
= \frac{1344 + 560 - 960 - 420}{105}
\]

\[
= \frac{524}{105}
\]

\[
\approx 4.99
\]
Double Integrals

Example: Integrate \( f(x,y) = 2y \) over \( R \) if \( R \) is the region bounded by \( x = y \) and \( x = (y-2)^2 \).

We will find it easier to slice along the \( x \)-direction first.

Limits on \( x \): from \( x = (y-2)^2 \) on left to \( x = y \) on right.

Limits on \( y \): from \( y = 1 \) to \( y = 4 \) (points of intersection).

\[
\int \int_R 2y \, dA = \int_1^4 \int_{(y-2)^2}^y 2y \, dx \, dy
\]

\[
= \int_1^4 2y \left[ \frac{x^2}{2} \right]_{(y-2)^2}^y \, dy
\]

\[
= \int_1^4 2y^2 - 2(y^2 - 4y + 4)y \, dy
\]

\[
= \int_1^4 2y^3 - 2y^5 + 8y^2 - 8y \, dy
\]

\[
= \left[ \frac{2y^4}{4} - \frac{2y^6}{6} + \frac{8y^3}{3} - 4y^2 \right]_1^4
\]

\[
= -\frac{1}{2} y^4 + \frac{10}{3} y^3 - 4y^2 \bigg|_1^4
\]

\[
= -128 + \frac{640}{3} - 64 - \left[ -\frac{1}{2} + \frac{10}{3} - 4 \right]
\]

\[
= -\frac{375}{2} + 210 = \frac{45}{2}
\]

This would be difficult to integrate in the opposite order since we would need two regions because the lower bound function changes at \( x = 1 \).
Double Integrals

Sometimes we are given an iterated integral and we want to change the order of integration.

\[
\int_0^{2x} \int_0^x f(x,y) \, dy \, dx
\]

1. Draw \( R \).
2. Label boundaries.
3. Transpose slice.

Integrate in \( x \) first \( \Rightarrow \) bounds are
\[ x = y/2 \text{ to } x = \sqrt{y} \]
Integrate in \( y \) second \( \Rightarrow \) bounds are \( 0 \leq y \leq 4 \).

\[
\int_0^4 \int_{y/2}^{\sqrt{y}} f(x,y) \, dx \, dy
\]

Ex. \( \int_0^1 \int_y^1 e^{x^2} \, dx \, dy \)

Limits on \( x \) are \( y \) to 1
Limits on \( y \) are 0 to 1.

Unfortunately we don't know the antiderivative of \( e^{x^2} \), so we cannot begin by finding it. Let's try reversing the order of integration to see if that helps...

Limits on \( y \) : 0 to \( x \)
Limits on \( x \) : 0 to 1

\[
\int_0^1 \int_0^x e^{x^2} \, dy \, dx = \int_0^1 y e^{x^2} \bigg|_0^x \, dx = \int_0^1 xe^{x^2} \, dx = \left[ \frac{1}{2} e^{x^2} \right]_0^1 = \frac{1}{2} e - \frac{1}{2} e^0 = \frac{1}{2} (e-1)
\]