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Question 1: Out of 4 people, how many ways can a president, a vice president and a treasurer be chosen?

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Question 1: Out of 4 people, how many ways can a president, a vice president and a treasurer be chosen?

Answer 1:

4 ways to choose the president.
3 ways to choose the vice president.
2 ways to choose the treasurer.

$4 \times 3 \times 2 = 24$ ways.

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Question 2: Out of 4 people, how many ways can 3 board members be chosen?

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Question 2: Out of 4 people, how many ways can 3 board members be chosen?

Answer 2: We can list the ways this can be done:

{1,2,3}, {1,2,4}, {1,3,4}, {2,3,4}

So there are 4 ways.

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Why are there different numbers of ways to do these two tasks?

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Why are there different numbers of ways to do these two tasks?

- In the first question the **order is important**.
- In the second question the **order is unimportant**.

The first question is a **permutation** problem while the second is a **combination** problem.

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The number of ***r*-permutations** of a set with n distinct elements is

$$P(n,r) = n(n-1)(n-2)...(n-r+1) = \frac{n!}{(n-r)!}$$

The number of ***r*-combinations** of a set with n distinct elements is

$$C(n,r) = \frac{n!}{r!\,(n-r)!}$$

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

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Notice that

$$C(n,r) = \frac{P(n,r)}{r!}$$

or

$$P(n,r) = C(n,r)\ r!$$

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

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Returning to our original two questions, suppose that after we chose the three board members in question 2 we counted the number of ways that they could be arranged. Let’s do this by listing all possible arrangements:

$$\{1,2,3\},\ \{2,3,1\},\ \{3,1,2\},\ \{1,3,2\},\ \{3,2,1\},\ \{2,1,3\}$$

We see that there are 6 arrangements. There are, of course, 6 arrangements for each of the 4 ways to choose the board members, yielding $6\times 4 = 24$ ways.

This is the same number of ways we found question 1 could be answered. Thus *choosing n elements where order matters can be done the same number of ways as choosing n elements without regard to order and then ordering them.*

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In question 1, since order matters, we use a permutation: $P(4,3) = 4\times 3\times 2 = 24$.

In question 2 order does no matter so we use a combination: $C(4,3) = 4!/(3!\times 1!) = (4\times 3\times 2\times 1)/(3\times 2\times 1\times 1) = 4$.

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For both $C(n,r)$ and $P(n,r)$ we take n to be a positive integer and r to be an integer such that $0 \leq r \leq n$.

Notice that $C(n,r) = C(n,n-r)$

because

$$C(n,n-r) = \frac{n!}{(n-r)!\,[n-(n-r)]!} = \frac{n!}{(n-r)!\,r!} = C(n,r)$$

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Examples:

$P(7,2) = 7!/(7-2)! = (7\times6\times5\times4\times3\times2\times1)/(5\times4\times3\times2\times1) = (7\times6) = 42$

$$
\begin{aligned}
C(7,2) &= 7!/[2!(7-2)!] \\
&= (7\times6\times5\times4\times3\times2\times1)/(2\times1\times5\times4\times3\times2\times1) \\
&= (7\times6)/(2\times1) \\
&= 21
\end{aligned}
$$

$$
\begin{aligned}
C(1024,1020) &= 1024!/[1020!(1024-1020)!] \\
&= (1024\times1023\times1022\times1021\times1020!)/[1020!\times(4\times3\times2\times1)] \\
&= 45545029376
\end{aligned}
$$

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Binomial Theorem

The **binomial theorem** uses combinations. In this context they are often called **binomial coefficients**.

Theorem: Let x and y be variables and let n be a positive integer. Then

$$(x+y)^n = C(n,0)x^n + C(n,1)x^{n-1}y + C(n,2)x^{n-2}y^2 + \ldots + C(n,n-1)xy^{n-1} + C(n,n)y^n.$$

Example

$$
\begin{aligned}
(3+2a)^4 &= C(4,0)(3^4) + C(4,1)(3^3)(2a) + C(4,2)(3^2)(2a)^2 + C(4,3)(3)(2a)^3 + C(4,4)(2a)^4 \\
&= 1\times81 + 4\times27\times2a + 6\times9\times4a^2 + 4\times3\times8a^3 + 1\times16a^4 \\
&= 81 + 216a + 216a^2 + 96a^3 + 16a^4
\end{aligned}
$$

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Pascal’s Triangle

One way to generate the binomial coefficients is with **Pascal's Triangle**.

```
                1
              1   1
            1   2   1
          1   3   3   1
        1   4   6   4   1
      1   5  10  10   5   1
    1   6  15  20  15   6   1
                      \   /
  1   7  21  35  35  21   7   1
1   8  28  56  70  56  28   8
```

Notice how each entry in the triangle is the sum of the two elements above it.

This relationship is described by Pascal's

Identity.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

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Pascal's Identity

Theorem: Let n and r be positive integers with $n>r$. Then

$$C(n,r) = C(n-1,r) + C(n-1,r-1)$$

Proof: Select a particular element t from the n elements in the set. When r elements are chosen from the set of n elements, either t will be in that subset or it will not. Thus we have two cases:

1. t is not in the subset: The number of elements to choose from is now n-1 since we exclude t, but we still need to chose r of them. This can be done $C(n-1,r)$ ways.

2. t is in the subset: Since t is already identified, we need only choose r-1 elements from a set of n-1 elements. This can be done $C(n-1,r-1)$ ways.

Applying the sum rule we conclude that $C(n,r) = C(n-1,r) +$

$C(n-1,r-1)$.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

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Permutations with Repetition

Question: An urn contains 4 red balls and 2 white balls. How many samples of two balls contain two red balls?

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

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Permutations with Repetition

Question: An urn contains 4 numbered red balls and 2 numbered white balls. How many samples of two balls contain two red balls?

Answer: We need to choose 2 of the 4 red balls. Since the balls are numbered it makes a difference which two we choose. We need to use a permutation; in this case $P(4,2) = 4 \times 3 = 12$.

Now suppose that once a ball is selected and the color is observed, it is replaced into the urn before the next selection. How many samples contain two red balls now?

$$4 \times 4 = 16$$

Theorem: The number of ***r*-permutations with repetition** of a set of n objects is n^r. This is also called an ***r*-sequence**.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

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Combinations with Repetition

Question: How many ways can five bills from a money drawer with seven denominations be selected?

In answering this note the following:

- We can repeatedly choose a particular denomination
- We can think of the money drawer as list of bills and separators:

$100 | $50 | $20 | $10 | $5 | $2 | $1

- If we "select" a bill by placing a marker in that denomination’s bin, then the problem at hand reduces to "How many ways can 5 markers and 6 separators be arranged?"

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

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Combinations with Repetition

Question: How many ways can five bills from a money drawer with seven denominations be selected?

For example, if we chose two $50 bills, a $10 bill, a $5 bill and a $2 bill the arrangement would look like

```
     | x x |     |  x  |  x  |  x  |
$100   $50   $20   $10   $5    $2    $1
```

These 11 items can be arranged $C(11,5)$ (or $C(11,6)$) different ways.

Notice that there are 6 separators because there are 7 denominations.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

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Combinations with Repetition

Theorem: There are $C(n+r-1,r)$ ***r*-combinations with repetition** from the set with n elements. These are also called ***r*-collections**.

Notice that $C(n+r-1,r) = C(n-1+r,n-1)$, a form also commonly used.

Example: A cookie shop has 9 varieties of cookies. How many different ways are there to select

- a dozen cookies? (answer)
- a dozen cookies with at least one of each type? (answer)

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

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Permutations of sets with indistinguishable objects

How many different 5-letter strings can be formed from the string ORONO?

Incorrect solution: $P(5,5) = 5! = 120$.

The problem with this approach is that it assumes each "O" is distinguishable from the other "O"s. So the strings

RONOO
RONOO
RONOO

are all counted as distinct strings.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

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Permutations of sets with indistinguishable objects

How many different 5-letter strings can be formed from the string ORONO?

The right way to go about this is to first place the three "O"s into the five available slots, then place the remaining letters.

- There are $C(5,3)$ ways to place the "O"s
- There are $C(2,1)$ ways to place the "R"
- There are $C(1,1)$ ways to place the "N"

Using the product rule we have $C(5,3)\times C(2,1)\times C(1,1) = 20$ ways.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

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Permutations of sets with indistinguishable objects

Theorem: The number of different permutations of n objects, where there are n_i indistinguishable objects of type i is

$$\frac{n!}{n_1!\, n_2! \ldots n_k!}$$

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

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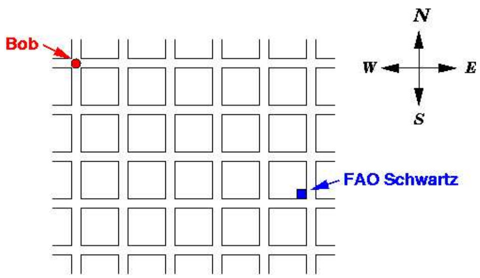
Permutations of sets with indistinguishable objects

This diagram shows a map of some city blocks. Bob is in the upper left corner and he wants to visit FAO Schwartz. Assuming that he only travels east (right) and south (down), how many possible paths does he have?



1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

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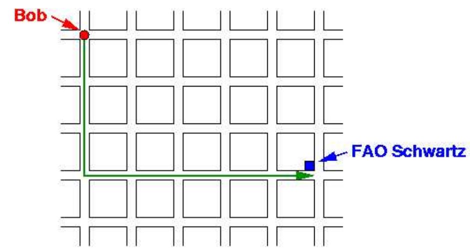
Permutations of sets with indistinguishable objects

Clearly, Bob has many choices. He could first travel south three blocks and then east five blocks:



This path could be diagrammed SSSEEEEE.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

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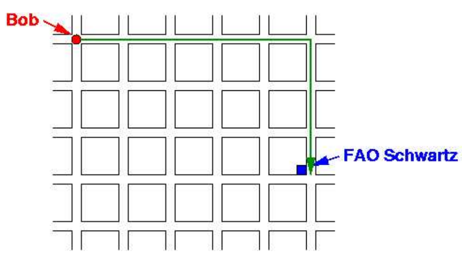
Permutations of sets with indistinguishable objects

Or, he could travel east five blocks and then south three blocks:



This path could be diagrammed: EEEEESSS.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

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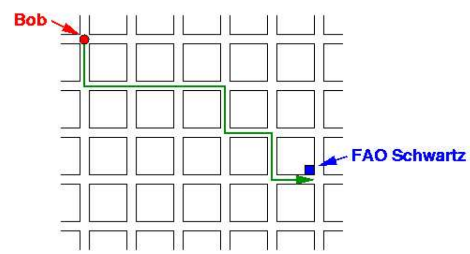
He could also wonder south and east an an almost arbitrary pattern:



This path could be diagrammed: SEEESESE.

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Combinations and Permutations

Permutations of sets with indistinguishable objects

The key here, and what makes this an example of permutations of indistinguishable objects, is that no matter what path Bob takes, he will move east five times and move south three times. Thus, the number of possible paths correspond to number of permutations of the string EEEEESSS.

There are 8 letters, including 5 E's and 3 S's. The number of permutations is given by $8!/(5!\times3!) = 56$. Thus, Bob has 56 different paths he could take.