prev | slides | next

Probability Theory

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17

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Probability Theory

prev | slides | next

We can redefine how we compute the probability of an event E in a way that lets us work with outcomes that are not equally likely.

The **probability of the event *E*** is the sum of the probabilities of the outcomes in E. If $E = \{a_1, a_2, ..., a_m\}$ then

$$p(E) = p(a_1) + p(a_2) + ... + p(a_m)$$

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17

Probability Theory

prev | slides | next

Example: A coin is tossed three times. What is the probability that a head comes up exactly once?

Solution: Let E = "a head occurs exactly once."

Old Method: $p(E) = |E|/|S| = C(3,1)/(2^3) = 3/8$.

New Method: $p(E) = p(\text{HTT}) + p(\text{THT}) + p(\text{TTH}) = 1/8 + 1/8 + 1/8 = 3/8$.

Not surprisingly, in the case of equally likely outcomes this these two calculations yield the same result.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17

Probability Theory

prev | slides | next

Example: Suppose that a die is loaded so that a 3 is twice as likely to appear as any other number. Then

$$p(3) = 2/7$$
$$p(1) = p(2) = p(4) = p(5) = p(6) = 1/7$$

This result is strange at first, but makes sense when one sets

$$a = p(1) = p(2) = p(4) = p(5) = p(6)$$
$$b = p(3) = 2a$$

so that $5a + b = 5a + 2a = 7a = 1$. This sum must be one because some number comes up on top when the die is rolled. It is easy to see that $a = 1/7$ so $b = 2/7$.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17

Probability Theory

prev | slides | next

As before we have that

- $p(E') = 1 - p(E)$
- $p(E \cup F) = p(E) + p(F) - p(E \cap F)$

i.e., these formulas do not depend on equally likely outcomes.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17

Probability Theory

prev | slides | next

Conditional Probability

Let E and F be events with $p(F) > 0$. The **conditional probability of *E* given *F***, denoted by $p(E|F)$, is defined by

$$p(E|F) = p(E \cap F) / p(F)$$

Example: A coin is tossed three times. What is the probability that it comes up heads all three times given that it comes up heads the first time?

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17

Probability Theory

prev | slides | next

Example: A coin is tossed three times. What is the probability that it comes up heads all three times given that it comes up heads the first time?

Solution:

E = "heads occurs three times," F = "heads occurs first time"

$$|S| = 2^3 = 8, \quad |E| = 1, \quad |F| = 4, \quad |E \cap F| = 1$$

$$p(E \cap F) = 1/8, \quad p(F) = 4/8 = 1/2$$

$$p(E|F) = (1/8) / (1/2) = 1/4.$$

This answer makes sense. If the first toss is known to be an H, we just need the next two out of two tosses to be H's, and the probability of that is 1/4.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17

Probability Theory

prev | slides | next

Independence

Notice that subsequent coin tosses are **independent** of one another.

Question: Suppose a fair coin has been tossed n times and come up heads each time. What is the probability that it comes up heads on the $n+1^{st}$ toss?

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17

Probability Theory

prev | slides | next

The events E and F are **independent** if and only if

$$p(E \cap F) = p(E)\, p(F)$$

which is equivalent to

$$p(E|F) = p(E) \text{ when } p(F) > 0.$$

This generalizes in the obvious way to more than two events:

$$p(E_1 \cap E_2 \cap \ldots \cap E_n) = p(E_1)\, p(E_2) \ldots p(E_n)$$

which is true if and only if all the events $E_1, E_2, \ldots E_n$ are independent.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17

Probability Theory

prev | slides | next

Bernoulli Trials

Some experiments have only two possible outcomes, e.g. tossing a coin; such experiments are called **Bernoulli Trials**. We can easily compute the probablity that one of these outcomes will occur a particular number of times in a sequence of trials.

The probability of k successes in n independent Bernoulli trials is

$$P_B = C(n,k)\, p^k\, q^{n-k}$$

where p is the probability of success and q=1-p.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17

Probability Theory

prev | slides | next

Example: A fair coin is tossed 50 times. What is the probability that 20 heads occur?

Solution:

$$p = 1/2, \qquad q = 1/2$$

$$P_B = C(50,20)\ 0.5^{20}\ 0.5^{30} = 0.04186$$

Example: A biased coin (probability of a head is 0.4) is tossed 50 times. What is the probability that 20 heads occur?

Solution:

$$P_B = C(50,20)\ 0.4^{20}\ 0.6^{30} = 0.11456$$

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17

Probability Theory

prev | slides | next

Example: A family has five children. Assume that the probability of a girl is p(G) = 0.49 and that the probability of a boy is p(B) = 0.51 and that the sexes of the children are indepenedent. What is the probability of the family having

a. exactly three boys?

b. at least one boy?

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17

Probability Theory

prev | slides | next

Example: A family has five children. Assume that the probability of a girl is $p(G) = 0.49$ and that the probability of a boy is $p(B) = 0.51$ and that the sexes of the children are indepenedent. What is the probability of the family having

a. exactly three boys?

Solution: We can use Bernoulli trials.

$$\begin{aligned} p(\text{exactly 3 boys}) &= C(5,3)\, p(B)^3\, p(G)^2 \\ &= 10\, (0.51)^3\, (0.49)^2 \\ &= 0.3185 \end{aligned}$$

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17

Probability Theory

prev | slides | next

Example: A family has five children. Assume that the probability of a girl is $p(G) = 0.49$ and that the probability of a boy is $p(B) = 0.51$ and that the sexes of the children are indepenedent. What is the probability of the family having

b. at least one boy?

Solution: We can again use Bernoulli trials. If we also compute the probability of the complementary event (no boys) then this problem is straight forward:

$$\begin{aligned} p(\text{at least 1 boy}) &= 1 - p(\text{no boys}) \\ &= 1 - C(5,0)\, p(B)^0\, p(G)^5 \\ &= 1 - (0.49)^5 \\ &= 0.9718 \end{aligned}$$

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17

Probability Theory

prev | slides | next

Random Variables and Expected Values

A **random variable** is a function from the sample space of an experiment to the set of real numbers.

Note that a random varible is not random and it is not a variable.

Example: Consider an experiment that consists of tossing a fair coin twice. We can define a random variable X such that

$$\begin{aligned} X(HH) &= 2 \\ X(HT) = X(TH) &= 1 \\ X(TT) &= 0 \end{aligned}$$

where here X specifies the number of heads.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17

Probability Theory

prev | slides | next

The **expected value** of the random variable $X(s)$ on the sample space $S=\{s_1, s_2, ..., s_n\}$ is equal to

$$E(X) = p(s_1)X(s_1) + p(s_2)X(s_2) + ... + p(s_n)X(s_n)$$

Example: Use the random variable from the last example and compute the expected value of X when a coin is tossed twice.

$$\begin{aligned} E(X) &= p(s_1)X(s_1) + p(s_2)X(s_2) + p(s_3)X(s_3) + p(s_4)X(s_4) \\ &= (1/4)\times 2 + (1/4)\times 1 + (1/4)\times 1 + (1/4)\times 0 \\ &= (1/4)(2 + 1 + 1 + 0) = 1 \end{aligned}$$

Since $E(X) = 1$ we expect that one head will come up when the coin is tossed twice.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17

Probability Theory

prev | slides | next

The expected value of the number of successes when n Bernoulli trials are performed is

$$E(X) = np$$

where p is the probability of success.

Example: What is the expected number of heads that come up when a fair coin is tossed five times?

Solution:

$$E(X) = 5\ (1/2) = 5/2$$

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17