PROOF EXAMPLES

MAT229 DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

Here are some examples of techniques used to prove implications in the form $p \to q$ are true. Recall that p is the hypothesis and q is the conclusion.

1. Trivial Proof

A trivial proof is one in which the conclusion is already known to be true. For example:

Theorem 1. *If $n \geq 0$ then $n^2 \geq 0$.*

Proof. Since n^2 is always a nonnegative integer when n is an integer, there is nothing to prove; the conclusion is already true. □

2. Direct Proof

Theorem 2. *If n is an integer greater than 1, then $n^2 > n$.*

Proof.

$$\begin{aligned} 1 &< n \\ &= n \cdot 1 \\ &< n \cdot n \\ &= n^2 \end{aligned}$$

Therefore $n < n^2$. □

Theorem 3. *If n is an odd integer, then $3n$ is an odd integer.*

Proof.

$$\begin{aligned} n &= 2q + 1 \text{ for some integer } q \\ 3n &= 6q + 3 \\ &= 2(3q) + 3 \\ &= 2(3q + 1) + 1 \end{aligned}$$

Since we've expressed $3n$ in the form $2k + 1$ where k is an integer we know that $3n$ is an odd integer. □

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3. Indirect Proof

Theorem 4. *If* $3n + 10 > 50$ *then* $n > 13$.

Proof. Assume $n \leq 13$. Then $3n \leq 39$ and so $3n+10 \leq 49$, which is the negation of the hypothesis (recall that $\neg q \rightarrow \neg p$ is logically equivalent to $p \rightarrow q$). □

Theorem 5. *If* n^2 *is even then* n *is even.*

Proof. Assume that n is odd. We will show that n^2 must also be odd.

If n is odd then there is an integer k such that $n = 2k + 1$. In this case

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned}$$

Since n^2 is written as an even number plus one, it must be an odd number.

We have just directly proved that "if n is odd then n^2 is odd" and thereby indirectly proved our theorem "if n^2 is even then n is even." □

4. Proof by Contradiction

Theorem 6. $\sqrt{2}$ *is an irrational number.*

Proof. Assume that $\sqrt{2}$ rational, in which case it can be expressed as the ratio of two integers with no factors in common:

$$\sqrt{2} = \frac{a}{b}$$

where a and b are both integers with no factors in common and with $b \neq 0$. Clearing the fraction by multiplying both sides by b and squaring gives

$$2b^2 = a^2$$

From this we see that a^2 must be an even number since it is a multiple of 2 and from one of the above examples we know that if a^2 is even then a is even. Thus we can write $a = 2c$ for some integer c. Our last equation then becomes

$$\begin{aligned} 2b^2 &= (2c)^2 \\ &= 4c^2 \\ b^2 &= 2c^2 \end{aligned}$$

Notice that the last line asserts that b^2 is an even number since it is a multiple of 2, and if b^2 is even then we know that b is even.

We now have a contradiction. We began by assuming that $\sqrt{2} = a/b$ where a and b had no common factors but we've just found that both a and b are even numbers, meaning they both have 2 as a factor. Since this is impossible, it must be that our assumption that $\sqrt{2}$ is rational is false. □

Theorem 7. *Let $g : A \to B$ and $f : B \to C$. If f and $f \circ g$ are both one-to-one functions then g is also a one-to-one function.*

Proof. Assume that g is not one-to-one. Then $\exists x_1, x_2 \in A$ such that $g(x_1) = g(x_2) = y$ for some $y \in B$. If $f(y) = z$ for some $z \in C$ then $(f \circ g)(x_1) = (f \circ g)(x_2) = z$, contradicting the fact that $f \circ g$ is one-to-one. Therefore it must not be possible that g is not one-to-one, and so g is indeed a one-to-one function. □

5. Proof by Cases

Theorem 8. $|x| + |y| \geq |x + y|$

Proof. **case 1:** $x \geq 0$, $y \geq 0$. In this case $|x + y| = x + y = |x| + |y|$.

case 2: $x \leq 0$, $y \leq 0$. Now $|x+y| = -(x+y) = -x+(-y) = |x|+|y|$.

case 3: $x \geq 0$, $y \leq 0$.

$$\begin{aligned} |x + y| &= ||x| - |y|| \\ &\leq ||x| + 2|y| - |y|| \\ &= ||x| + |y|| \\ &= |x| + |y| \end{aligned}$$

case 4: $x \leq 0$, $y \geq 0$. Here we merely need to reverse the roles of x and y in case 3.

Since all possible cases have been handled, and in each case we see that $|x| + |y| \geq |x + y|$, we have prove the theorem. □