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Let A and B be sets.

The **union** of A and B is the set of all elements in either A or B or in both sets. It is denoted $A \cup B$.

The **intersection** of A and B is the set of all elements in both A and B. It is denoted $A \cap B$.

$$A \cup B = \{x \mid x \in A \lor x \in B\}$$
$$A \cap B = \{x \mid x \in A \land x \in B\}$$

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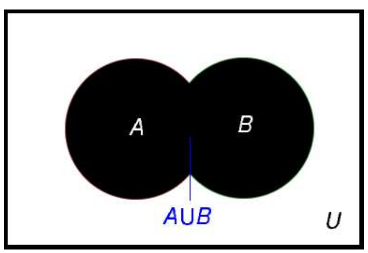
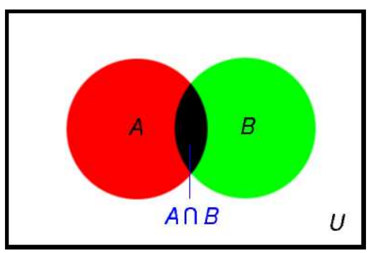
In terms of Venn diagrams we have

Union



Intersection



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Two sets are **disjoint** if their intersection is the empty set.

If $A=\{1, 2\}$ and $B=\{3, 4, 5\}$ then they are disjoint because $A \cap B = \varnothing$.

Question: how big is $A \cup B$?

For example, suppose $A=\{1, 2, 3, 4\}$ and $B=\{2, 4, 6\}$. In this case

$$A \cup B=\{1,2,3,4,6\}$$

$$|A| = 4 \quad |B| = 3 \quad |A \cup B| = 5$$

How can we figure out what the cardinality of $A \cup B$ directly from the cardinality of A and the cardinality of B?

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In fact, we cannot calculate $|A \cup B|$ knowing only $|A|$ and $|B|$. We need one additional piece of information.

The difficulty lies in that if we find $|A|+|B|$ we've counted the elements that are common to both sets twice. We need some way to reduce this by the number of elements common to both sets...

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The **inclusion-exclusion principle** comes to our rescue. In its most basic form it is

$$|A \cup B| = |A| + |B| - |A \cap B|$$

The last term subtracts the sum by the number of elements common to both sets, which is exactly what we needed to do.

Suppose $|A \cup B| = 7$, $|A| = 5$ and $|A \cap B| = 3$. What is $|B|$?

(answer)

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Let A and B be sets. The **difference** of A and B, denoted A-B, is the set containing all elements that are in A but not also in B. This set is also called the **complement of *B* with respect to *A***.

$$A-B = \{x \mid x \in A \wedge x \notin B\}$$

If U is the universal set then the **complement of *A*** is denoted A' and is the set U-A.

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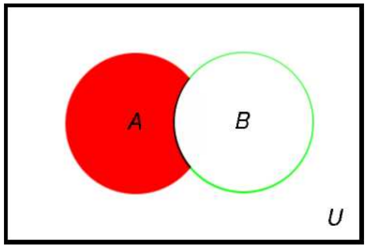
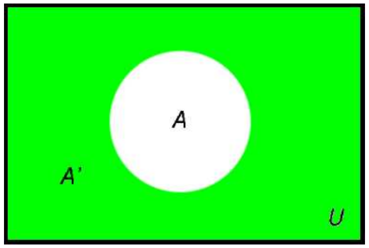
Here are Venn diagrams for these two types of complements:

A-B



A'



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Set Identities

Identity	Name
$A \cup \varnothing = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \varnothing = \varnothing$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$(A')' = A$	Complementation law

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Set Identities

Identity	Name
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$(A \cup B)' = A' \cap B'$ $(A \cap B)' = A' \cup B'$	De Morgan's laws

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Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ using a **common element proof**.

$$
\begin{aligned}
x \in A \cap (B \cup C) &\equiv x \in A \wedge x \in (B \cup C) \\
&\equiv x \in A \wedge (x \in B \vee x \in C) \\
&\equiv (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C) \\
&\equiv x \in (A \cap B) \vee x \in (A \cap C) \\
&\equiv x \in (A \cap B) \cup (A \cap C)
\end{aligned}
$$

We have assumed that x is in $A \cap (B \cup C)$ and shown that in this case x must be in $(A \cap B) \cup (A \cap C)$. Thus we have that

$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$$

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The bad news is that we've only completed half of the proof: to show equality we need still to show that

$$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$$

The good news is that this is easy. Because each of the steps in the first part of the proof is logically equivalent to the step before it, it follows that if we start at the bottom and assume that x is in $(A \cap B) \cup (A \cap C)$ then we can conclude that x is in $A \cap (B \cup C)$ and we are done.

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The main idea behind a common element proof to show that two sets A and B are equal is to first assume that x is a typical element of A and show that it must be in B. This shows that

$$A \subseteq B.$$

Next, assume that x is in B and show that it must also be in A. This shows that

$$B \subseteq A$$

Finally, because $A \subseteq B$ and $B \subseteq A$ it must be that $A = B$.

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