2.4 Differences Between Linear and Nonlinear Equations

Questions of <u>Existence and uniqueness</u>

1. Does a solution exist?
2. If so, is it unique?
3. For what values of the independent variable does the solution exist?

Theorem 2.4.1

If $p(x)$ and $g(x)$ are <u>continuous</u> functions on the open interval $I: \alpha < x < \beta$ containing the point $x = x_0$, then <u>there exists</u> a <u>unique</u> function $\phi(x)$ that satisfies the differential equation

$$y' + p(x)y = g(x)$$

for each x in I, and that also satisfies the initial condition

$$y(x_0) = y_0$$

where y_0 is an arbitrary prescribed initial value.

<u>Proof</u>: Since $p(x)$ is continuous on I, we know that $\int p(x)dx$ exists and so $\mu(x)$ is defined and continuous on I.

$\mu(x)$ is nonzero and differentiable.

$$\mu(x)y' + \mu(x)p(x)y = (\mu(x)y)' = \mu(x)g(x).$$

But $\mu(x)g(x)$ is integrable since both μ and g are continuous, thus we know

$$y = \frac{1}{\mu(x)}\left[\int \mu(x)g(x)\,dx + c\right] \qquad (*)$$

exists. Now lets make sure this is a solution:

$$y' = \frac{-\mu'(x)}{\mu(x)^2}\left[\int \mu(x)g(x)\,dx + c\right] + \frac{1}{\mu(x)}\mu(x)g(x)$$

but $\mu(x) = e^{\int p(x)\,dx}$ so $\mu'(x) = e^{\int p(x)\,dx}p(x)$
$= \mu(x)p(x)$

so $$y' = -\frac{\mu(x)p(x)}{\mu(x)^2}\left[\int \mu(x)g(x)\,dx + c\right] + g(x)$$

$$y' + p(x)y = -\frac{p(x)}{\mu(x)}\left[\int \mu(x)g(x)\,dx + c\right] + g(x) + p(x)\frac{1}{\mu(x)}\left[\int \mu(x)g(x)\,dx + c\right]$$

$$= [0]\left[\int \mu(x)g(x)\,dx + c\right] + g(x)$$

$$= g(x) \checkmark$$

Thus the solution does exist. Now, about uniqueness...

Since the integral in (*) is indefinite, it has a constant of integration associated with it. Of course, we can express this as a definite integral and by choosing x_0 for the lower bound we get

$$\int \mu(x)g(x)\,dx = \int_{x_0}^{x} \mu(s)g(s)\,ds + C$$

Where c depends on x_0. So

$$y = \frac{1}{\mu(x)}\left[\int_{x_0}^{x} \mu(s)\, g(s)\, ds + c\right]$$

Recall that $\mu(x) = e^{\int p(x)dx}$ or $e^{\int_{x_0}^{x} p(s)ds}$ since the choice of the constant of integration here is arbitrary.
Thus $\mu(x_0) = e^{\int_{x_0}^{x_0} p\, dx} = e^0 = 1$.

$$\text{So} \quad y(x_0) = \frac{1}{\mu(x_0)}\left[\int_{x_0}^{x_0} \mu(s)\, g(s)\, ds + c\right]$$

$$= \frac{1}{1}\left[0 + c\right] = c$$

but $y(x_0) = y_0$ so $c = y_0$ and we have

$$y = \frac{1}{\mu(x)}\left[\int_{x_0}^{x} \mu(s)\, g(s)\, ds + y_0\right]$$

$$\text{where} \quad \mu(x) = e^{\int_{x_0}^{x} p(x)\, dx}.$$

Thus, for any particular initial condition, only one solution exists $\Rightarrow$ uniqueness.

Ex Find the solution of $\begin{cases} x^2 y' - xy = \frac{1}{x} \\ y(1) = -1 \end{cases}$

and determine for what values of x the solution is valid.

1. put DE into standard form: $y' + py = g$

$$y' - \frac{1}{x} y = \frac{1}{x^3} \qquad p(x) = -\frac{1}{x} \qquad g(x) = \frac{1}{x^3}$$

2. Find $\mu(x) = \exp \int \left(-\frac{1}{x}\right) dx = \exp[-\ln x] = 1/x$

3. $y = \frac{1}{\mu} \int \mu g \, dx = x \int \frac{1}{x} \frac{1}{x^3} dx = x\left[\left(-\frac{1}{3x^3}\right) + c\right]$

$$= -\frac{1}{3x^2} + cx$$

4. $y(1) = -\frac{1}{3} + c = -1$

$$c = -2/3$$

So the solution is $\boxed{y(x) = -\frac{1}{3x^2} - \frac{2x}{3}}$

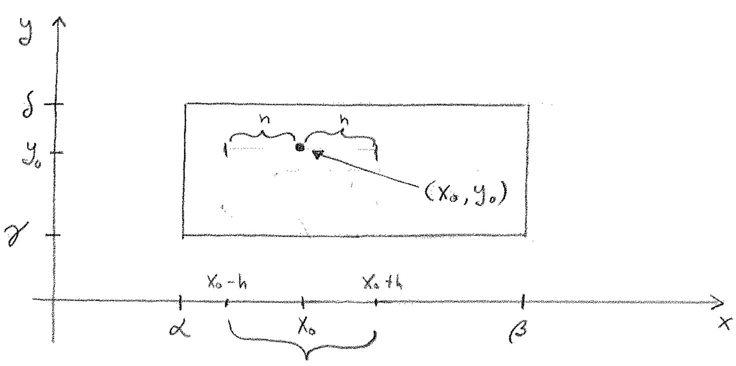
This solution is defined for $x \neq 0$ and if the solution is to be continuous $x > 0$.

2.4 Linear vs. Nonlinear Equations

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases} \qquad *$$

Theorem 2.4.2

Let the functions f and $\frac{\partial f}{\partial y}$ be continuous in some rectangle $\alpha < x < \beta$, $\gamma < y < \delta$ containing the point (x_0, y_0). Then, in some interval $x_0 - h < x < x_0 + h$ contained in $\alpha < x < \beta$, there is a unique solution $y = \phi(x)$ of the initial value problem $(*)$.



interval in which the solution $y = \phi(x)$ is guarenteed to exist uniquely.

Ex #11 $\quad \frac{dy}{dx} = \frac{1+x^2}{3y-y^2}$ $\quad$ Does this satisfy the cond. of the theorem?

$$f(x,y) = \frac{1+x^2}{3y-y^2}$$ is continuous if $3y - y^2 \neq 0$

$y(3-y) \neq 0$ so $y \neq 0, y \neq 3$

$$\frac{\partial f}{\partial y} = -\frac{1+x^2}{(3y-y^2)^2}(3-2y) = -\frac{(1+x^2)(3-2y)}{(3y-y^2)^2}$$

again, cont. if $y \neq 0, y \neq 3$

$\boxed{y \neq 0, y \neq 3}$

TEXT EXAMPLE #3 $\quad y' = y^{1/3}$, $\quad y(0) = 0$

$x \geq 0$

Separable →

$$y^{-1/3}dy = dx$$

$$\frac{3}{2} y^{2/3} = x + C$$

$$y^{2/3} = \frac{2}{3}(x+C)$$

$$y = \pm\left[\frac{2}{3}(x+C)\right]^{3/2}$$

Using $y(0)=0$

$$0 = \pm\left[\frac{2}{3}(0+C)\right]^{3/2} \rightarrow C = 0$$

So

$$y = \begin{cases} \phi_1(x) = \left(\frac{2}{3}x\right)^{3/2} \\ \phi_2(x) = -\left(\frac{2}{3}x\right)^{3/2} \end{cases}$$

Note: Solution is not unique for any $x > 0$. In fact, $y = \phi_3(x) = 0$ is also a solution!

According to text - when I.C. is $y(x_0) = 0$

$$y = \chi(x) = \begin{cases} 0 & 0 \leq x < x_0 \\ \pm\left[\frac{2}{3}(x - x_0)\right]^{3/2} & x \geq x_0 \end{cases}$$

for all positive x_0.

- Continuity : ok
- differentiability

$$\frac{d}{dx}\left[\frac{2}{3}(x-x_0)\right]^{3/2} = \frac{3}{2}\left[\frac{2}{3}(x-x_0)\right]^{1/2}\left[\frac{2}{3}\right] = \sqrt{\frac{2}{3}(x-x_0)}$$

yes, and deriv is cont. when $x = x_0$ since $0 = 0$,

ok

What is going on here?

$f(x,y) = y^{1/3}$ is cont for all y

$\frac{\partial}{\partial y} f(x,y) = \frac{d}{dy} y^{1/3} = \frac{1}{3} y^{-2/3} = \frac{1}{3y^{2/3}}$

which is not defined when $y=0$

Thus, the theorem fails when γ and δ have opposite sign or if one is zero.

However, if $y_0 \neq 0$, then $y' = y^{1/3}$, $y(x_0) = y_0$ does have a unique solution which passes through (x_0, y_0)

Ex #13 $\frac{dy}{dx} = -4\frac{x}{y}$ $y(0) = y_0$

Solve $y\,dy = -4x\,dx$

$\frac{y^2}{2} = -2x^2 + C'$

$y^2 = 2(C' - 2x^2)$

$y^2 = C - 4x^2$

$y = \pm\sqrt{C - 4x^2}$

apply i.c.

$y_0 = \pm\sqrt{C - 0} = \pm\sqrt{C} \rightarrow C = y_0^2$

Soln: $y = \pm\sqrt{y_0^2 - 4x^2}$. Need $y_0^2 \geq 4x^2$ $\boxed{|x| \leq \frac{|y_0|}{2}}$

$|y_0| \geq 2|x|$

Not quite done – what if $y_0 = 0$?

$$0 = \pm\sqrt{C} \Rightarrow C = 0 \quad \text{so} \quad y = \pm\sqrt{-4x^2}$$

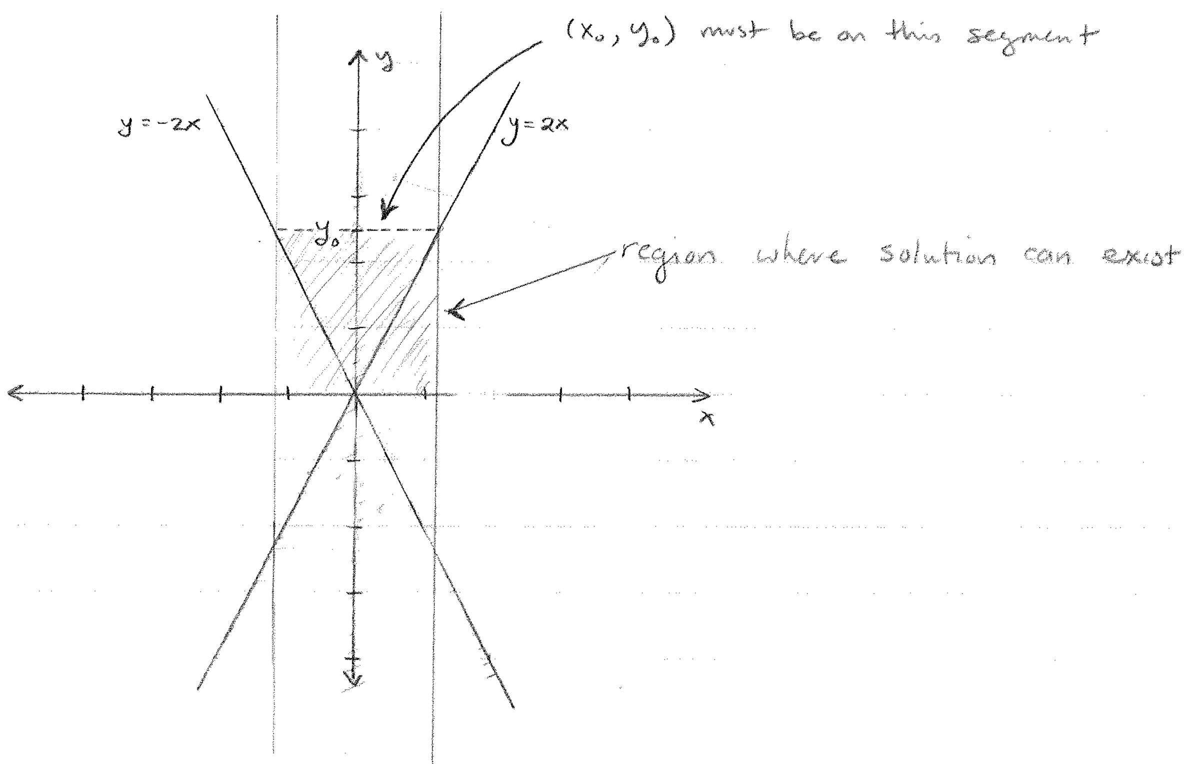
Valid only if $x = 0$

$$\therefore \quad y = \pm\sqrt{y_0^2 - 4x^2}$$

if $y_0 \neq 0$ and $|x| \leq \frac{|y_0|}{2}$

Is it unique? For a particular y_0 yes, it is.

$$y = \begin{cases} \sqrt{y_0^2 - 4x^2} & \text{if } y_0 > 0 \\ -\sqrt{y_0^2 - 4x^2} & \text{if } y_0 < 0 \end{cases}$$



Ex #14 $\quad y' = 2xy^2 \qquad y(0) = y_0$

$$y^{-2}dy = 2x\,dx$$
$$-y^{-1} = x^2 + c$$
$$y = -\frac{1}{x^2+c}$$

now apply I.C.

$$y(0) = y_0 = -\frac{1}{0+c} = -\frac{1}{c} \Rightarrow c = -\frac{1}{y_0} \quad \text{if } y_0 \neq 0$$

So $\quad y = -\dfrac{1}{x^2 - \frac{1}{y_0}} = \dfrac{y_0}{1 - x^2 y_0}$

So we need $\quad 1 - x^2 y_0 \neq 0$

$$\frac{1}{y_0} \neq x^2 \quad \Rightarrow x \neq \pm\sqrt{\frac{1}{y_0}} \text{ if } y_0 > 0$$

if $y_0 = 0$ then we see from our orig. eq that $y = 0$
Since if $y_0 = 0$ then initially $y' = 0$ so y does not change

Thus: If $y_0 = 0 \rightarrow \quad y = 0 \quad \forall x$

If $y_0 \neq 0 \rightarrow \quad y = \dfrac{y_0}{1 - y_0 x^2}$

with $-\sqrt{\frac{1}{y_0}} < x < \sqrt{\frac{1}{y_0}} \qquad$ if $y_0 > 0$

and $-\infty < x < \infty \qquad$ if $y_0 < 0$