Autonomous Equations and Population Dynamics

The first order ode $\frac{dy}{dt} = f(y)$ is called <u>autonomous</u>.
The RHS depends on y alone - not on the independent variable t.

We have seen several of these - most significantly $y' = ry$, the exponential growth/decay equation.

One interesting fact about autonomous equations is that their direction fields are easy to draw because f does not depend on t.

Exponential Growth : $y' = ry$

- simple model that works well for populations with no impediments to growth
- of course, growth cannot occur indefinitely

Logistic Growth.

Let's modify the exponential growth eq. so that

1. $y' \approx ry$ when y is small
2. $y' \to 0$ as $y \to K$ where K is the population that can be sustained.

$$y' = ry\underbrace{(1 - y/K)}$$

- close to 1 when y is close to zero
- close to 0 when y is close to K
- positive if $y < K$, negative if $y > K$

This gives the logistic growth equation

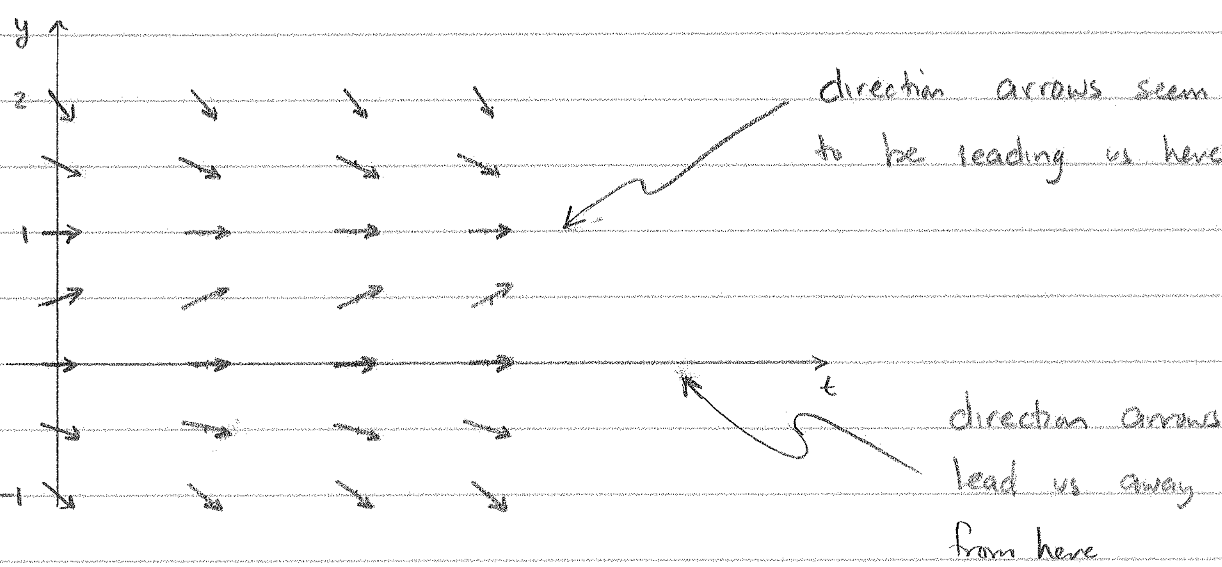
$$\boxed{\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y}$$

r is the intrinsic growth rate

K is the saturation level or environmental carrying capacity

Ex. $\frac{dy}{dt} = \frac{1}{2}\left(1 - \frac{y}{1}\right)y = \frac{1}{2}(1-y)y$ $\quad r = \frac{1}{2}$, $K = 1$

The direction field for this equation looks like



Consider $\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$, $\quad r > 0, \ K > 0$

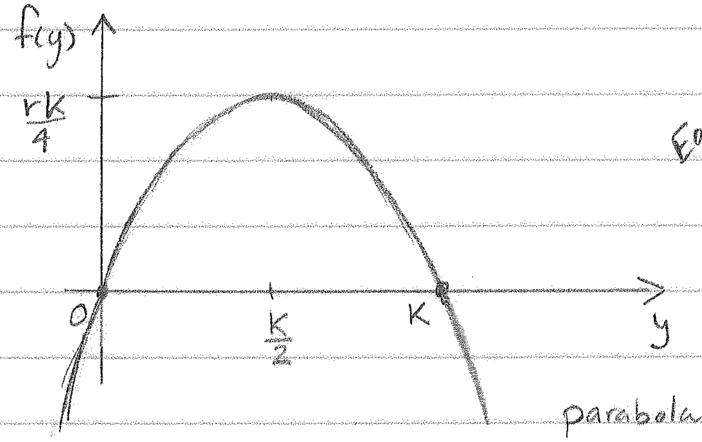
If $y > 0$ and $y < K$ then $r(1 - y/K) > 0 \Rightarrow y$ increases

If $y > 0$ and $y > K$ then $r(1 - y/K)y < 0 \Rightarrow y$ decreases

So, the solution $y(t)$ will approach K as $t \to \infty$

Autonomous Equations and Population Dynamics

Given $\frac{dy}{dt} = f(y) = r(1 - y/K)y$ we can plot $f(y)$ vs. y



Equilibrium points $\frac{dy}{dt} = 0$ (Same as critical points)

$\frac{dy}{dt} < 0$

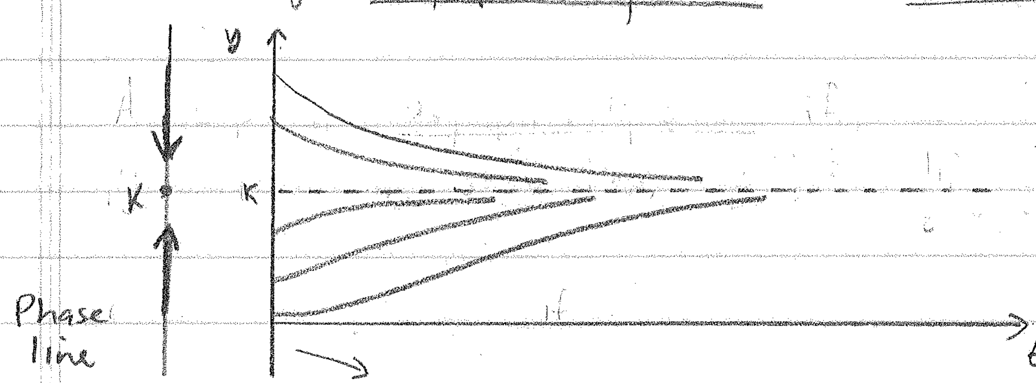
Values of y for which dy/dt is positive, meaning $y(t)$ is increasing

Values of y for which $\frac{dy}{dt}$ is negative meaning $y(t)$ is decreasing

This is why K is called the <u>saturation level</u>: If y is above it, y will decrease towards it. If y is below it, y will increase towards it.

The points 0 and K are called <u>critical points</u>. equilibrium points

The critical points in these problems can be characterized as being <u>asymptotically stable</u> or <u>unstable</u>



To solve the logistic equation $\frac{dy}{dt} = r(1-y/K)y$ we separate variables

$$\frac{dy}{(1-y/K)y} = r\,dt$$

To integrate the LHS we can use partial fractions

$$\frac{1}{(1-y/K)y} = \frac{A}{1-y/K} + \frac{B}{y} \implies A = 1/K, \quad B = 1$$

So $\frac{dy}{K(1-y/K)} + \frac{dy}{y} = r\,dt$

$$\int \frac{dy}{K-y} + \int \frac{dy}{y} = \int r\,dt$$

$$-\ln|K-y| + \ln|y| = rt + c$$

$$\ln\left|\frac{y}{K-y}\right| = rt + c$$

$$\frac{y}{K-y} = Ce^{rt}$$

If $y(0) = y_0$ then $\frac{y_0}{K-y_0} = C$ so $\frac{y}{K-y} = \frac{y_0}{K-y_0}e^{rt}$

Solving for y $\frac{1}{\frac{K}{y}-1} = \frac{y_0}{K-y_0}e^{rt} \implies \frac{K}{y} - 1 = \frac{K-y_0}{y_0}e^{-rt}$

$$\frac{K}{y} = \frac{(K-y_0)e^{-rt} + y_0}{y_0} \implies \frac{y}{K} = \frac{y_0}{y_0 + (K-y_0)e^{-rt}}$$

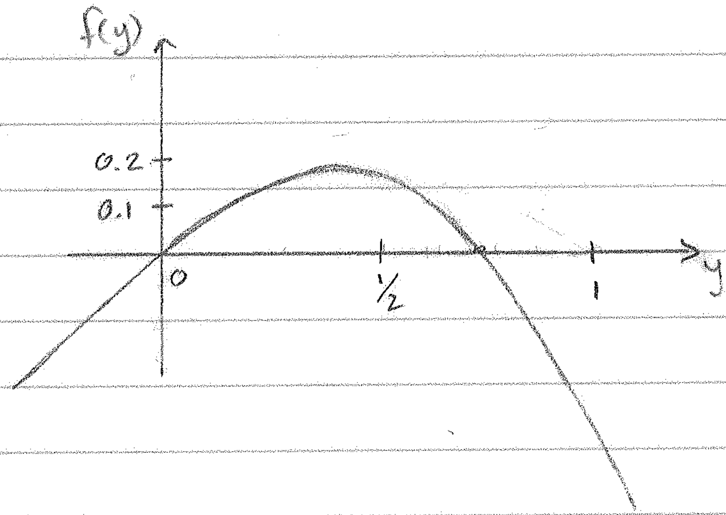
$$\boxed{y = \frac{Ky_0}{y_0 + (K-y_0)e^{-rt}}}$$

Notice that as $t \to \infty$ $y \to Ky_0/y_0 = K$ (if $r > 0$).

Autonomous Equations and Population Dynamics

Ex Let $\frac{dy}{dt} = f(y)$ be $\frac{dy}{dt} = 2y(1 - \frac{e^y}{2})$ $\quad -\infty < y_0 < \infty$

and sketch $f(y)$ vs. y. Find the critical points and classify them as asymptotically stable or unstable



Critical points at

$0 = 2y(1 - \frac{1}{2}e^y)$

so $y = 0$ is one.

$0 = 1 - \frac{1}{2}e^y$

$2 = e^y$

$y = \ln 2 \simeq 0.6931$

is the other

$y = 0$ unstable critical point since $f(y) > 0$ when $y > 0$, $f(y) < 0$ when $y < 0$

$y = \ln 2 \approx 0.6931$ asymptotically stable critical point since $f(y) > 0$ when $y < \ln 2$ and $f(y) < 0$ when $y > \ln 2$.