3.5 Repeated Roots; Reduction of Order

Consider $y''-2y'+y=0 \rightarrow r^2-2r+1=0$

$(r-1)(r-1)=0$ so $r=1$ is a repeated root.

Clearly $y_1=e^x$ is a solution. But $y_2=e^x$ is the same – i.e. y_1 and y_2 do not form a linearly independent set of solutions – the wronskian of y_1 and y_2 is zero.

Let us try another approach: if e^x is a solution, we already know that ce^x is also a solution. Perhaps $v(x)e^x$ will also be a solution – we can put it into the o.d.e. and find what $v(x)$ needs to be.

$$y=v(x)e^x \qquad y'=v'e^x+ve^x=(v'+v)e^x$$

$$y''=v''e^x+v'e^x+v'e^x+ve^x$$

$$=(v''+2v'+v)e^x$$

So

$$(v''+2v'+v)e^x-2(v'+v)e^x+ve^x=0$$

$$v''e^x+(2v'-2v')e^x+(v-2v+v)e^x=0$$

$$v''e^x=0$$

$$v''=0$$

$$v'=c_1$$

$$v=c_1x+c_2$$

So we find that $y=(c_1x+c_2)e^x=c_1xe^x+c_2e^x$ is a solution.

Actually, this is our <u>general solution</u>, the fundamental solution set being xe^x and e^x.

Given $ay'' + by' + cy = 0$

Let $y = e^{rt}$ so $ar^2 + br + c = 0$

If $r = \alpha, \alpha$ is a repeated root then $y_1 = e^{\alpha t}$ is a solution. Let $y = V(t)e^{\alpha t}$

$y' = V'e^{\alpha t} + \alpha V e^{\alpha t}$ $\quad$ $y'' = V''e^{\alpha t} + 2\alpha V' e^{\alpha t} + \alpha^2 V e^{\alpha t}$

$$ay'' + by' + cy = a(V'' + 2\alpha V' + \alpha^2 V)e^{\alpha t} + b(V' + \alpha V)e^{\alpha t} + cVe^{\alpha t} = 0$$

$$= V'' a e^{\alpha t} + V'(2\alpha a + b)e^{\alpha t} + V(a\alpha^2 + b\alpha + c)e^{\alpha t} = 0$$

$a\alpha^2 + b\alpha + c = 0$ since $r = \alpha$ is a root.

If a quadratic eq has repeated roots it must have a horizontal tangent at $y = 0$ so $\frac{d}{dr}(ar^2 + br + c) = 0$ at $r = \alpha$: $\left.\frac{d}{dr}\right|_\alpha (ar^2 + br + c) = 2ar + b\Big|_\alpha = 2a\alpha + b = 0$

$\therefore$ $V'' a e^{\alpha t} = 0$. We require $a \neq 0$ so $V'' = 0$

$\Rightarrow V = c_1 t + c_2$

$$y = (c_1 t + c_2)e^{\alpha t} = c_1 \underbrace{te^{\alpha t}}_{\text{new, linearly independent solution}} + c_2 \underbrace{e^{\alpha t}}_{\text{already known}}$$ is also a soln.

To make sure, $W = \begin{vmatrix} xe^x & e^x \\ (x+1)e^x & e^x \end{vmatrix} = xe^{2x} - (x+1)e^{2x} = e^{2x} \neq 0$

General Rule: If we get repeated roots of the characteristic eq., say $r=\alpha$, then a fundamental set of solutions is $e^{\alpha x}$ and $xe^{\alpha x}$.

So, given $ay'' + by' + cy = 0$:

① Solve characteristic eq. $ar^2 + br + c = 0$ for roots r_1 and r_2

② Case 1: $r_1 \neq r_2$, real

$\rightarrow y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

Case 2: $r_1 = \lambda + i\mu$, $r_2 = \lambda - i\mu$

$\rightarrow y = c_1 e^{\lambda x} \cos \mu x + c_2 e^{\lambda x} \sin \mu x$

Case 3: $r_1 = r_2 = r$

$\rightarrow y = c_1 x e^{rx} + c_2 e^{rx}$

Notice that both cases 1 and 3 involve exponentials which can go to zero or to infinity. Note that relatively few, if any, sign changes occur in the solution as $x \to \infty$.

This is in contrast to case 2 which has periodic functions as part of the solution.

Reduction of Order

If we know one solution of $y'' + p(x)y' + q(x)y = 0$ we can find another independent solution using a similar method – if y_1 is a solution then seek another solution of the form $v(x)\,y_1$.

y_1 satisfies $y_1'' + py' + qy = 0$

$y_2 = v y_1 \qquad y_2' = v'y_1 + vy_1' \qquad y_2'' = v''y_1 + 2v'y_1' + vy_1''$

ode becomes $\quad v''y_1 + 2v'y_1' + vy_1'' + p(v'y_1 + vy_1') + q(vy_1) = 0$

$$v''y_1 + v'(2y_1' + py_1) + v(\underbrace{y_1'' + py_1' + qy_1}_{0}) = 0$$

$$v''y_1 + v'(2y_1' + py_1) = 0 \qquad (*)$$

If $w = v'$

$P(x) = 2\frac{y_1'}{y_1} + p(x) \quad \Rightarrow \quad w' + P(x)\,w = 0$

$$\frac{dw}{w} = -P(x)\,dx$$

$$\ln|w| = -\int P(x)\,dx$$

$$w = v' = e^{-\int 2\frac{y_1'}{y_1} + p\,dx}$$

integrate to find $v(x)$

note: non constant coefficients

Ex $\quad x^2y'' + 2xy' - 2y = 0 \qquad x > 0 \qquad y_1(x) = x$

$y'' + \frac{2}{x}y' - \frac{2}{x^2}y = 0 \qquad y_2(x) = v(x)\,x$

$p(x) = 2/x \qquad y_2' = v'x + v$

$q(x) = -2/x^2 \qquad y_2'' = v''x + v' + v' = v''x + 2v'$

from $(*)$ $\quad v''x + v'(2\cdot 1 + \frac{2}{x}x) = 0 \Rightarrow xv'' + 4v' = 0$

$$w = v' \quad \to \quad x w' = -4w$$

$$\frac{dw}{w} = -4\,\frac{dx}{x}$$

$$\ln|w| = -4\ln|x| + c$$

$$w = c\,x^{-4}$$

$$v = \int c x^{-4}\,dx = c_1 x^{-3} + c_2$$

So $\boxed{y_2 = x \cdot x^{-3} = x^{-2} \text{ is also a soln.}}$

Check:

$$x^2(6x^{-4}) + 2x(-2x^{-3}) - 2x^{-2} = 6x^{-2} - 4x^{-2} - 2x^{-2} = 0 \checkmark$$

general solution would be $\boxed{y = c_1 x + c_2 x^{-2}}$

For discussion #33, 34

Critically Damped Harmonic Oscillator

$m\ddot{x} + \gamma\dot{x} + kx = 0$ $\quad\gamma$ is coefficient of friction, $\gamma \geq 0$

Assume $x = e^{rt} \Rightarrow mr^2 + \gamma r + k = 0 \to r = \dfrac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}$

if $\gamma^2 > 4km \to$ two real distinct roots: $x = c_1 e^{\frac{-\gamma - \sqrt{\gamma^2 - 4km}}{2m}t} + c_2 e^{\frac{-\gamma + \sqrt{\gamma^2 - 4km}}{2m}t}$

exponents are both negative so solutions decay exponentially

if $\gamma^2 < 4km \to$ complex conjugate roots $r = \frac{-\gamma}{2m} \pm \frac{\sqrt{4km - \gamma^2}}{2m}\,i = \lambda + i\mu$

$x = c_1 e^{\lambda t}\cos\mu t + c_2 e^{\lambda t}\sin\mu t$; decays since $\lambda < 0$

if $\gamma^2 - 4km = 0 \to$ real repeated roots $r = -\gamma/2m$

$x = c_1 e^{-\frac{\gamma}{2m}t} + c_2 t e^{-\frac{\gamma}{2m}t}$ Decay, no oscillation –

Slowest decay w/o osc. $\longrightarrow$ Critically damped