4.3 Method of Undetermined Coefficients

If $L[y] = g(x)$, and if L has constant coefficients we can use the method of undetermined coefficients if $g(x)$ is of an appropriate form.

$g(x)$ must be a sum and/or product of

1. polynomials of x
2. exponentials in x
3. sines and cosines in x

① Polynomial part. If $g(x)$ is a polynomial of degree m then we can assume a particular solution of the form $Y(x) = A_0 x^m + A_1 x^{m-1} + \dots + A_m$.

Problem:

$$y^{iv} - y'' = x \quad \rightarrow \quad r^4 - r^2 = 0 \quad \rightarrow \quad r = 0, 0, 1, -1$$

$$\rightarrow \quad y_c = C_1 + C_2 x + C_3 e^x + C_4 e^{-x}$$

Now, since $g(x) = x$, we assume a particular solution of the form $Y(x) = A_1 x + A_0$. However, this is part of the general solution - y^{iv} and y'' are both zero.

Maybe $Y(x) = A_1 x^2 + A_0 x$? No - still doesn't give x.

Try $Y(x) = A_1 x^3 + A_0 x^2$

$$0 - 6A_1 x - A_0 = x \quad \rightarrow \quad A_1 = -\frac{1}{6}, \quad A_0 = 0$$

So, if $r=0$ is an s-fold root of the characteristic eq. [s is the order of the lowest derivative of y in the eq], then $Y(x) = x^s [A_0 x^m + A_1 x^{m-1} + \dots + A_m]$ is a part. soln.

Now consider exponentials of x. If $g(x) = P_m(x)\, e^{\alpha x}$
where $P_m(x)$ is a polynomial of degree m. We assume
a solution of the form

$$Y(x) = e^{\alpha x}\left[A_0 x^m + A_1 x^{m-1} + \cdots + A_m\right]$$

Which is fine unless α is a root of the characteristic eq.

$$y^{iv} - y'' = xe^x \quad \to \quad r = 0, 0, 1, -1 \quad \to \quad y_c = c_1 + c_2 x + c_3 e^x + c_4 e^{-x}$$

So we can't assume $Y = [A_0 x + A_1]e^x$ since part of
this is a solution of the homogeneous eq.

Try $Y(x) = x[A_0 x + A_1]e^x = A_0 x^2 e^x + A_1 x e^x$

$$Y' = A_0 x^2 e^x + 2A_0 x e^x + A_1 x e^x + A_1 e^x$$
$$Y'' = A_0 x^2 e^x + 4A_0 x e^x + 2A_0 e^x + A_1 x e^x + 2A_1 e^x$$
$$Y''' = A_0 x^2 e^x + 6A_0 x e^x + 6A_0 e^x + A_1 x e^x + 3A_1 e^x$$
$$Y^{iv} = A_0 x^2 e^x + 8A_0 x e^x + 12A_0 e^x + A_1 x e^x + 4A_1 e^x$$

$$Y^{iv} - Y'' = 4A_0 x e^x + 10A_0 e^x + 2A_1 e^x = x e^x$$

$$4A_0 = 1 \qquad 10A_0 + 2A_1 = 0$$
$$A_0 = 1/4 \qquad 10/4 = -2A_1 \to A_1 = -5/4$$

$Y = \frac{1}{4}[x^2 - 5x]e^x$ is a particular soln.

Now if $g(x) = P_m(x)\, e^{\alpha x}[K_1 \sin\beta x + K_2 \cos\beta x]$

We assume $Y(x) = [A_0 x^m + A_1 x^{m-1} + \dots + A_m]\, e^{\alpha x} \sin\beta x$
$+ [B_0 x^m + B_1 x^{m-1} + \dots + B_m]\, e^{\alpha x} \cos\beta x$

but will need additional factors of x if $\alpha \pm i\beta$ are roots of the homogeneous eq.

See table on top of page 204 5th, 218 6th, 175 7th, 181 8th

Method of Annihilators

We will use $D = \frac{d}{dx}$. Then $y'' - y = (D^2 - 1)y$
$y''' + 3y'' + 3y = (D^3 + 3D^2 + 3)y$

Ex. $y''' - 2y'' + y' = x^3 + 2e^x$

$$(D^3 - 2D^2 + D)y = D(D^2 - 2D + 1)y = \underbrace{D(D-1)^2 y = x^3 + 2e^x}$$

Suppose we knew what operators to apply to so that the RHS became zero, giving us a homogeneous eq.

How do we get $x^3 \to 0$? use D^4
$2e^x \to 0$? use $D - 1$

So, $D^4(D-1)$ will "annihilate" $x^3 + 2e^x$

$$D^4(D-1)(x^3+2e^x) = (D^5-D^4)(x^3+2e^x)$$
$$= D^5x^3 - D^4x^3 + 2D^5e^x - 2D^4e^x$$
$$= 0 - 0 + 2e^x - 2e^x = 0 \checkmark$$

$$\therefore \quad D^4(D-1)\cdot D(D-1)^2 y = 0$$
$$\underbrace{D^5(D-1)^3}_{\text{characteristic polynomial}} y = 0$$

$$r^5(r-1)^3 = 0 \rightarrow r = 0,0,0,0,0,1,1,1$$

So, $y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4 + c_6 e^x + c_7 x e^x + c_8 x^2 e^x$

complementary soln.

particular soln

Note that this method exchanges a nonhomogeneous differential equation for a higher order homogeneous equation. — This new equation is not necessarily any easier to solve...

Ex $y^{IV} + 4y'' = \sin 2x + xe^x + 4$

$D^2(D^2+4)y = \sin 2x + xe^x + 4$

find annihilators for $\sin 2x$, xe^x, 4

$\sin 2x$: D^2+4

xe^x : $(D-1)^2$

4 : D

So

$D(D^2+4)(D-1)^2 D^2(D^2+4)y = D(D^2+4)(D-1)^2(\sin 2x + xe^x + 4)$

$D^3(D^2+4)^2(D-1)^2 y = 0$

$r^3(r^2+4)^2(r-1)^2 = 0 \rightarrow r = 0, 0, 0, \pm 2i, \pm 2i, 1, 1$

$y(x) = c_1 + c_2 x + c_3 x^2 + c_4 \sin 2x + c_5 \cos 2x + c_6 x \sin 2x + c_7 x \cos 2x$
$+ c_8 e^x + c_9 x e^x$

Char. eq. of orig eq. is $r^2(r^2+4) = 0 \rightarrow r = 0, 0, \pm 2i$

so 1, x, $\sin 2x$, $\cos 2x$ are solns of homogeneous problem so...

$Y(x) = A_1 x^2 + x(B_1 \sin 2x + B_2 \cos 2x) + (C_1 + C_2 x)e^x$

is a particular soln.

Find A_1, B_1, B_2, C_1, C_2 by substituting this back into the original equation