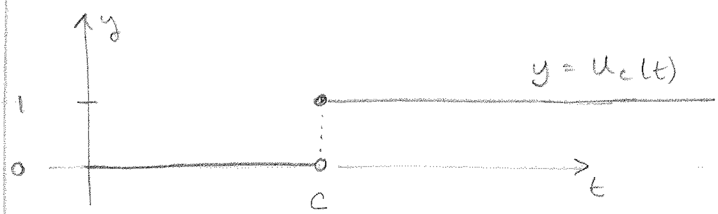
6.3 Step Functions

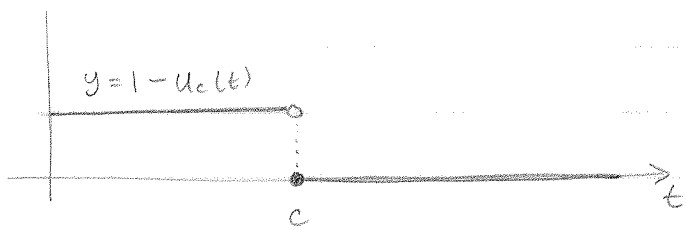
The <u>unit step function</u>

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases} \qquad c \geq 0$$

allows us to "turn on" or "turn off" functions at will.

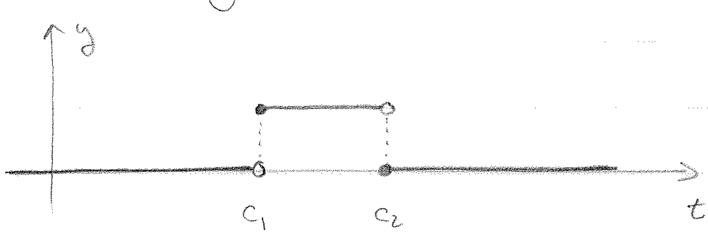


Turn on



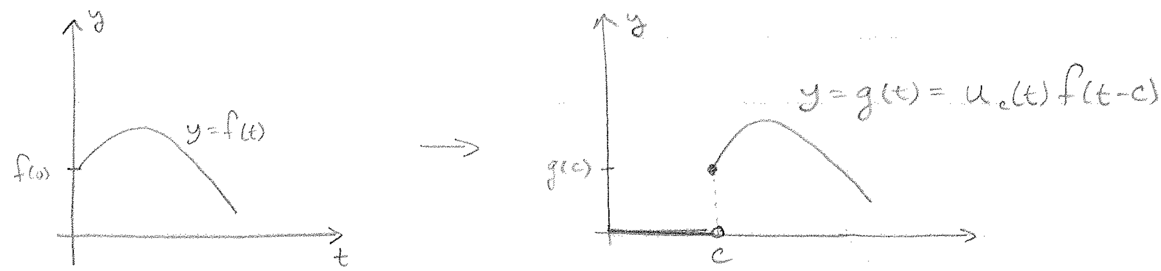
Turn off

How about $y = u_{c_1}(t) - u_{c_2}(t)$, $c_1 \leq c_2$



Now, Suppose we have a function $f(t)$, $t \geq 0$, and want to translate it to the right an amount c. If we also want our new function to be zero from $t=0$ to $t=c$ we can accomplish all this with

$$g(t) = \underbrace{u_c(t)}_{\text{zero until } t=c} \underbrace{f(t-c)}_{\text{shift to the right}}$$



What is $\mathcal{L}\{g(t)\}$?

$$\mathcal{L}\{g(t)\} = \int_0^\infty e^{-st} g(t)\,dt = \int_0^\infty e^{-st} u_c(t) f(t-c)\,dt$$

$$= \int_c^\infty e^{-st} f(t-c)\,dt$$

$$\xi = t-c \rightarrow d\xi = dt \qquad \xi\Big|_{t=c} = 0$$

so $$\mathcal{L}\{g(t)\} = \int_0^\infty e^{-s(\xi+c)} f(\xi)\,d\xi = e^{-cs}\int_0^\infty e^{-s\xi} f(\xi)\,d\xi$$

$$= e^{-cs}\mathcal{L}\{f(\xi)\} = e^{-cs}\mathcal{L}\{f(t)\}$$

Thm 6.3.1

If $F(s) = \mathcal{L}\{f(t)\}$, $s > a \geq 0$, $c > 0$ then

$$\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs}F(s) \qquad s > a$$

and

$$u_c(t) f(t-c) = \mathcal{L}^{-1}\{e^{-cs}F(s)\}$$

Ex Find $\mathcal{L}\{f(t)\}$ if $f(t) = \begin{cases} 0 & 0 \le t < 1 \\ t & 1 \le t \end{cases}$

(need $f(t)$ to appear as a function of $t-c$ ...)

$$f(t) = u_1(t)\, t = u_1(t)(t-1) + u_1(t)$$

$$\text{so } \mathcal{L}\{f(t)\} = e^{-s}\mathcal{L}\{t\} + e^{-s}\mathcal{L}\{1\} = e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right)$$

Ex Find $\mathcal{L}\{f(t)\}$ if $f(t) = \begin{cases} 4 & 0 \le t < 2 \\ t^2 & 2 \le t \end{cases}$

$$f(t) = 4 + u_2(t)[t^2 - 4] = 4 + u_2(t)\,[(t-2)(t+2)]$$

$= (t-2)(t-2+4)$
$= (t-2)^2 + 4(t-2)$

$$= 4 + u_2(t)\,[(t-2)^2 + 4(t-2)]$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{4\} + e^{-2s}\mathcal{L}\{t^2 + 4t\}$$

$$= \frac{4}{s} + e^{-2s}\left[\frac{2}{s^3} + \frac{4}{s^2}\right]$$

Ex Find $\mathcal{L}^{-1}\{F(s)\}$ if $F(s) = (1 - e^{-\pi s})\frac{s}{s^2+4}$

$$F(s) = \frac{s}{s^2+4} - e^{-\pi s}\frac{s}{s^2+4}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \mathcal{L}^{-1}\left\{e^{-\pi s}\frac{s}{s^2+4}\right\}$$

$$= \cos 2t - u_\pi(t)\cos 2(t-\pi)$$

don't forget the shift

$$= \cos 2t - u_\pi(t)\left[\cos 2t \cos 2\pi + \sin 2t \sin 2\pi\right]$$

$$= \cos 2t - u_\pi(t)\cos 2t$$

$$= [1 - u_\pi(t)]\cos 2t$$

Thm 6.3.2

If $F(s) = \mathcal{L}\{f(t)\}$, $s > a \geq 0$ then

$$\mathcal{L}\{e^{ct} f(t)\} = F(s-c) \qquad s > a+c$$

and

$$e^{ct} f(t) = \mathcal{L}^{-1}\{F(s-c)\}$$

Ex Find $\mathcal{L}^{-1}\{F(s)\}$ if $F(s) = \dfrac{2(s-1)\,e^{-2s}}{s^2-2s+2}$

$$F(s) = \frac{2(s-1)e^{-2s}}{s^2-2s+1\ +1} = \frac{2(s-1)e^{-2s}}{(s-1)^2+1}$$ (complete the square)

$$= 2e^{-2s} \frac{s-1}{(s-1)^2+1}$$

$$\frac{s-1}{(s-1)^2+1} \xrightarrow{\mathcal{L}^{-1}} e^t \cos t$$

need to shift by 2 because of e^{-2s} so

$$\boxed{\mathcal{L}^{-1}\{F(s)\} = 2\,u_2(t)\,e^{t-2}\cos(t-2)}$$